THE CONNECTED COMPONENTS OF PARTIALLY HYPERBOLIC ENDOMORPHISMS ON $\mathbb{T}^2$

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Motivation. One of the main goals of dynamical systems is the classification of different dynamics, and in particular, which are the canonical models to compare with. This goal was completely achieved in the uniformly hyperbolic setting for invertible maps on the torus $\mathbb{T}^n$ (actually for nilmanifolds) where it was shown that every Anosov diffeomorphism on the torus $\mathbb{T}^n$ is conjugated to its linear part, which is in fact a hyperbolic matrix [Fr], [Fr1] & [Man].

In the partially hyperbolic setting this classification relies on the existence of a center foliation. However, there are examples of partially hyperbolic systems having a non-integrable center bundle: in the *invertible* case examples of this type are constructed in [RHRHU] & [Wi], and in the *non-invertible* case examples are built in [HH1], [HH2] & [HYS] (in fact these last examples are inspired in [RHRHU]).

In the past years, a series of papers have shown that for *invertible* partially hyperbolic systems, the integrability (or not) of the center distribution is a property that depends directly on the isotopy class of the map and the topology of the manifold. On the one hand in [BFFP], [FPS], [Mar] & [PS], there are entire isotopy classes of partially hyperbolic diffeomorphisms having center foliations, and on the other hand, in [BFFP] the authors showed the existence of entire isotopy classes of partially hyperbolic diffeomorphisms with no center foliation at all. We want to remark that this phenomenon is quite surprising, since the integrability of the center bundle is usually related to some geometric assumptions, and on the other hand the isotopy class of a map is a purely topological property.

The purpose of this article is to investigate the structure of connected components of partially hyperbolic endomorphisms on $\mathbb{T}^2$ in the non-invertible setting. In particular, we identify dynamical properties that are shared by all partially hyperbolic endomorphisms within the same connected component.

Basic definitions. In this paper by endomorphism we mean a non-invertible C^1 local diffeomorphism. We say that an endomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ is *partially hyperbolic* if it admits an unstable cone family $\mathcal{C}^u$. That is, there is a map $p \mapsto \mathcal{C}^u(p)$ and a constant $\lambda > 1$ such that $Df_p(v^u)$ belongs to the interior of $\mathcal{C}^u(f(p))$, and $\|Df_p(v^u)\| \geq \lambda \|v^u\|$, for every $v^u \neq 0$ in $\mathcal{C}^u(p)$. We are going to denote by $\mathcal{PH}(\mathbb{T}^2)$ to the set of all partially hyperbolic endomorphisms. We would like to point out that $\mathcal{PH}(\mathbb{T}^2)$ is an open set in C^1-topology.

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Every $f \in \mathcal{PH}(\mathbb{T}^2)$ necessarily admits a *center bundle* E^c_f, which is Df-invariant and transversal to $\mathcal{C}^u_f$. It can be defined by

$$E^c_f(x) := \bigcap_{n\geq 0} Df^{-n}(\mathcal{C}^c(f^n(x))),$$

where $\mathcal{C}^c$ is the *complementary cone* of $\mathcal{C}^u$, $p \mapsto \mathcal{C}^c(p) := T_p\mathbb{T}^2 \backslash int(\mathcal{C}^u_p)$, which is preserved by Df^{-1} (see [CP] for more details). Since E^c_f is one-dimensional, we always have curves tangent to E^c_f by Peano's theorem. Moreover, we also have a *center branching foliation*, which is an f-invariant collection of C^1 curves tangent to E^c_f which cover $\mathbb{T}^2$ and do not topologically cross ([BI, HH3]). However, we do not necessarily have a true foliation as the examples in [HH1], [HH2] & [HYS] show.

We say that $f \in \mathcal{PH}(\mathbb{T}^2)$ is *dynamically coherent* if there is an invariant foliation of $\mathbb{T}^2$, denoted by $\mathcal{W}^c_f$, which is tangent to the center bundle. By *foliation*, we mean a partition of $\mathbb{T}^2$ by flow lines tangent to a continuous vector field without singularities.

Once dynamical coherence is achieved, we can compare the behavior of different systems modulo center leaves: we say that two dynamically coherent partially hyperbolic endomorphisms f and g are *leaf conjugate* if there is a homeomorphism $h : \mathbb{T}^2 \to \mathbb{T}^2$ which sends f-center leaves to g-center leaves, and satisfies

$$h(\mathcal{W}^c_f(f(p))) = \mathcal{W}^c_g(g(h(p))),\ \forall p \in \mathbb{T}^2.$$

Therefore, leaf conjugacy is the analogous of conjugacy in the partially hyperbolic setting.

We should also point out that a partially hyperbolic endomorphism does not necessarily admit a global unstable bundle. When such a bundle exists, we refer to the map as *specially partially hyperbolic* (although this notion will not be treated specifically in this work). Nevertheless, any lift of the map to $\mathbb{R}^2$ admits a well-defined splitting $\mathbb{R}^2 = E^c \oplus E^u$, where E^u is an unstable bundle on $\mathbb{R}^2$. Moreover, this unstable bundle integrates uniquely to an invariant foliation, which we denote by $\widetilde{\mathcal{W}}^u$.

It is well known that every $f \in \mathcal{PH}(\mathbb{T}^2)$ is homotopic to a unique linear endomorphism $A : \mathbb{T}^2 \to \mathbb{T}^2$ which is induced by a 2×2 invertible matrix with integer coefficients. We call A the *linear part* or the *linearisation* of f. By a slightly abuse of notation, we are going to denote by A to both the linear endomorphism induced in $\mathbb{T}^2$, and the 2×2 matrix.

Now if $\widetilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the lift of f in the universal cover we known that $\widetilde{f}(\widetilde{p} + v) = \widetilde{f}(\widetilde{p}) + A(v)$, for every $\widetilde{p} \in \mathbb{R}^2$ and $v \in \mathbb{Z}^2$. In particular this implies

$$\operatorname{dist}(\widetilde{f}, A) = \sup_{\widetilde{p}\in\mathbb{R}^2} \|\widetilde{f}(\widetilde{p}) - A(\widetilde{p})\| < +\infty \tag{0.1}$$

and in the same way

$$\begin{aligned} \operatorname{dist}(\widetilde{f}^{-1}, A^{-1}) &= \sup_{\widetilde{p}\in\mathbb{R}^2} \|\widetilde{f}^{-1}(\widetilde{p}) - A^{-1}(\widetilde{p})\| \leq \|A^{-1}\| \sup_{\widetilde{p}\in\mathbb{R}^2} \|A \circ \widetilde{f}^{-1}(\widetilde{p}) - \widetilde{p}\| \\ &= \|A^{-1}\| \sup_{\widetilde{q}\in\mathbb{R}^2} \|A(\widetilde{q}) - \widetilde{f}(\widetilde{q})\| = \|A^{-1}\| \operatorname{dist}(\widetilde{f}, A). \end{aligned} \tag{0.2}$$

Denoting by λ_1 and λ_2 the two eigenvalues of A, we know that A can be classified into the following three types based on its eigenvalues:

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- if $|\lambda_1| < 1 < |\lambda_2|$, we say A is *hyperbolic*;
- if $1 = |\lambda_1| < |\lambda_2|$, we say A is *non-hyperbolic*;
- if $1 < |\lambda_1| \leq |\lambda_2|$, we say A is *expanding*.

Notice that if $|\lambda_1| \neq |\lambda_2|$ then A is partially hyperbolic itself and therefore it makes sense to talk about leaf conjugacy between f and A. We denote by $\mathcal{PH}_A(\mathbb{T}^2)$ the open set of all partially hyperbolic endomorphisms of $\mathbb{T}^2$ with linear part A. We also fix the notation $\mathcal{A}^c$ and $\mathcal{A}^u$ for the *linear foliations* (that is, foliations by parallel lines) generated by the eigendirections of A associated to the eigenvalues λ_1 and λ_2, respectively.

Previous results. In [HH1], it is proved that if A is hyperbolic then every $f \in \mathcal{PH}_A(\mathbb{T}^2)$ is dynamically coherent and leaf conjugate to A. This implies, in particular, that every connected component of $\mathcal{PH}_A(\mathbb{T}^2)$ contains only endomorphisms which are dynamically coherent and leaf conjugate to A.

On the other hand, when A is non-hyperbolic or expanding, there are examples of partially hyperbolic endomorphisms which are not dynamically coherent [HH1, HH2] & [HYS] (they are called *dynamically incoherent* examples). Moreover, in [HH3], the authors introduce a mechanism to obtain dynamically incoherent examples based on the existence of what they called a "periodic center annulus", which we now define.

An open set of $\mathbb{T}^2$ which is homeomorphic to $(-1,1)\times\mathbb{S}^1$ will be called by *annulus* and denoted by $\mathbb{A}$. We say that $f \in \mathcal{PH}(\mathbb{T}^2)$ admits a *periodic center annulus* if there is an annulus $\mathbb{A}$ whose boundary consists of either one or two C^1 circles tangent to E^c_f and $f^n(\mathbb{A}) = \mathbb{A}$ for some $n \geq 1$. Additionally, one requires $\mathbb{A}$ to be minimal with respect to these properties. We define the *period of* $\mathbb{A}$ as the smallest integer $n \geq 1$ such that $f^n(\mathbb{A}) = \mathbb{A}$.

Roughly speaking, in order to get the dynamically incoherent examples, the authors build a partially hyperbolic endomorphism exhibiting a periodic center annulus with branching behavior on the boundary leaves, preventing the integrability of the center bundle. This kind of annulus is built as the limit of foliated annulus displaying one of the following feature:

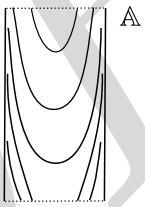

FIGURE 1. Reeb component.

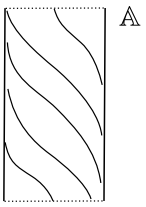

FIGURE 2. Tannulus.

These figures are taken from [AR].

For a set $X \subseteq \mathbb{R}^2$ and $R > 0$, we define a R-neighborhood of X in $\mathbb{R}^2$ by

$$U(X,R) = \left\{\widetilde{x} \in \mathbb{R}^2 : \operatorname{dist}(\widetilde{x}, X) < R\right\}.$$

Now, following [FPS] we introduce next definition which basically compares the asymptotic direction of two foliations.

Definition 0.1 (Almost parallel foliations)**.** *We say that two foliations $\widetilde{\mathcal{F}}_1$ and $\widetilde{\mathcal{F}}_2$ in $\mathbb{R}^2$ are almost parallel, if there is $R > 0$ such that, for every $\widetilde{x} \in \mathbb{R}^2$ there are points $\widetilde{x}_1, \widetilde{x}_2 \in \mathbb{R}^2$ such that:*

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- $\widetilde{\mathcal{F}}_1(\widetilde{x}) \in U(\widetilde{\mathcal{F}}_2(\widetilde{x}_1), R)$ *and* $\widetilde{\mathcal{F}}_2(\widetilde{x}_1) \in U(\widetilde{\mathcal{F}}_1(\widetilde{x}), R)$.
- $\widetilde{\mathcal{F}}_2(\widetilde{x}) \in U(\widetilde{\mathcal{F}}_1(\widetilde{x}_2), R)$ *and* $\widetilde{\mathcal{F}}_1(\widetilde{x}_2) \in U(\widetilde{\mathcal{F}}_2(\widetilde{x}), R)$.

This condition can be stated using the Hausdorff distance: for every $\widetilde{x} \in \widetilde{M}$, there is $\widetilde{x}_1, \widetilde{x}_2 \in \widetilde{M}$ such that $d_H(\widetilde{\mathcal{F}}_1(\widetilde{x}), \widetilde{\mathcal{F}}_2(\widetilde{x}_1)) < R$ and $d_H(\widetilde{\mathcal{F}}_2(\widetilde{x}), \widetilde{\mathcal{F}}_1(\widetilde{x}_2)) < R$. Additionally, the relation of being almost parallel is an equivalence relation.

In [Po, Proposition 4.A.2], we have the following.

Proposition 0.2. *If $\mathcal{F}$ is a foliation on $\mathbb{T}^2$ without Reeb components, then its lift $\widetilde{\mathcal{F}}$ to the universal cover is almost parallel to a linear foliation.*

Recall that every $f \in \mathcal{PH}(\mathbb{T}^2)$ has center branching folations which may or may not present periodic center annulus. We say that f presents no periodic center annulus if every center branching foliation for f presents no such annulus. In [HH3], assuming the absence of such periodic center annulus, the authors provide a comprehensive description of partially hyperbolic endomorphisms on $\mathbb{T}^2$. We summarize the main results in the following way.

Theorem 0.3 ([HH3])**.** *Let $f \in \mathcal{PH}_A(\mathbb{T}^2)$. If f does not admit a periodic center annulus then:*

- *A has eigenvalues λ_1 and λ_2 with $|\lambda_1| \neq |\lambda_2|$.*
- *f is dynamically coherent and leaf conjugate to A.*
- *Every invariant center branching foliation is almost parallel to the linear center foliation of A.*

Although [HH3] does not state explicitly the local uniqueness of the center bundle, it follows from their arguments. In particular, they establish a property of u-expansion: two distinct points on the same unstable leaf in the universal cover must separate in the π^u-direction under forward iterates of a lift $\widetilde{f}$.

If the center bundle were not locally unique at a point p, one could find two distinct forward center curves through p intersected by the same unstable leaf at points x_1 and x_2. By [HH3], their π^u-projections remain uniformly close under iteration, contradicting u-expansion. Hence the center bundle must be locally unique.

Another very nice characterization in [HH3] is the following.

Theorem 0.4 ([HH3])**.** *Let $f \in \mathcal{PH}_A(\mathbb{T}^2)$ be admitting periodic center annuli.*

- *If A is non-hyperbolic linearization then $\mathbb{T}^2$ is the union of the closures of all periodic center annuli of f.*
- *If A is expanding and $\mathbb{A}_1, \dots, \mathbb{A}_n$ the collection of all disjoint periodic center annuli of f. Then,*

$$\Lambda = \bigcup_i \bigcup_{k \geq 0} f^{-k}(\mathbb{A}_i)$$

is dense in $\mathbb{T}^2$. Moreover, there exists a unique semiconjugacy $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ from f to A such that if $\mathbb{A}$ is a connected component of Λ, then $\mathbb{A}$ is an annulus, and $\varphi(\mathbb{A})$ is either a periodic or preperiodic circle under A.

We would like to point out that the existence of a periodic center annulus $\mathbb{A}$ for f implies that the homotopy class of the boundary of $\mathbb{A}$, which is in $\pi_1(\mathbb{T}^2) = \mathbb{Z}^2$, is an eigenvector of A. Consequently, the eigenvalues λ_1 and λ_2 of A are integers.

In particular, if A has irrational eigenvalues then every $f \in \mathcal{PH}_A(\mathbb{T}^2)$ does not admit a periodic center annulus and, hence, it is dynamically coherent and leaf

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conjugate to A. Therefore, every connected component of $\mathcal{PH}_A(\mathbb{T}^2)$ contains only but endomorphisms which are dynamically coherent and leaf conjugate to A.

Main results. n light of the previous discussion, in this paper we focus on connected components of $\mathcal{PH}_A(\mathbb{T}^2)$, where A is either an expanding or a non-hyperbolic linear map with integer eigenvalues.

Our main results are the following.

Theorem A. *Let $f \in \mathcal{PH}_A(\mathbb{T}^2)$. If f is dynamically coherent and leaf conjugate to A, then every partially hyperbolic endomorphism in the same connected component of f is also dynamically coherent and leaf conjugate to A.*

The proof of Theorem A follows directly from the following result.

Theorem B. *Being dynamically coherent and leaf conjugate to A is a C^1 open and closed property among partially hyperbolic endomorphisms. More precisely,*

- *(I) If $f \in \mathcal{PH}_A(\mathbb{T}^2)$ is dynamically coherent and leaf conjugate to A then there is a C^1 neighborhood $\mathcal{U}$ of f such that every $g \in \mathcal{U}$ is dynamically coherent and leaf conjugate to A.*
- *(II) If $\{f_n\}_n \subseteq \mathcal{PH}_A(\mathbb{T}^2)$ and $f \in \mathcal{PH}_A(\mathbb{T}^2)$ are such that $f_n \to f$ in C^1-topology and each f_n is dynamically coherent and leaf conjugate to A then f is dynamically coherent and leaf conjugate to A.*

Recall that, in Theorem 0.3, it is proved that if f does not admit a periodic center annulus then f is dynamically coherent and leaf conjugate to its linearisation. Here, we establish the following result concerning the existence of periodic center annuli.

Theorem C. *The absence of periodic center annulus is shared by all partially hyperbolic endomorphism in the same connected component in $\mathcal{PH}_A(\mathbb{T}^2)$. Consequently, the existence of periodic center annuli is also a shared property within the same connected component.*

Our next result establishes that the number of periodic center annuli is constant in each connected component in $\mathcal{PH}_A(\mathbb{T}^2)$. Consequently, $\mathcal{PH}_A(\mathbb{T}^2)$ has infinitely many connected components.

Theorem D. *If f and g are in the same connected component of $\mathcal{PH}_A(\mathbb{T}^2)$ admitting periodic center annuli then both f and g have the same number of periodic center annuli.*

Finally, we remark that, by Theorem A, some connected components of $\mathcal{PH}_A(\mathbb{T}^2)$ consist entirely of endomorphisms that are dynamically coherent and leaf conjugate to A. Among these components, by Theorem C, some are formed by maps that do not admit periodic center annuli. In the remaining components, the maps do admit periodic center annuli, while still being dynamically coherent and leaf conjugate to A. The components without periodic center annuli are associated with local unique integrability of the center bundle, while in the other components such uniqueness fails.

We next construct some example of endomorphisms which is dynamically coherent and leaf conjugate to A, but nevertheless admit periodic center annuli.

Theorem E. *There exists $f \in \mathcal{PH}_A(\mathbb{T}^2)$ admits periodic center annuli while still being dynamically coherent and leaf-conjugate to A.*

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We present two examples, the first with A non-hyperbolic and the second with A expanding.

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