Resume

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On the paper [1] we study the one dimensional nonlinear wave equation

$$\begin{cases} \Box u = K(x)u^3 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \in (H^1 \times L^2) \end{cases} \tag{1}$$

where $\Box = -\partial_t^2 + \partial_x^2$. The localized function K is assumed to be smooth, nonnegative, compactly supported in $[-1, 1]$, with $K(0) > 0$, and to satisfy the repulsivity condition

$$xK'(x) \leq 0.$$

Although the underlying one-dimensional linear wave equation has no dispersive decay, the localized defocusing interaction produces enough spacetime control to obtain scattering and, moreover, allows one to recover the coefficient K from the scattering operator.

We begin by establishing a global well-posedness result for the model under consideration:

Theorem 0.1 (–, Campos, Murphy (2026)) *Let $K \in L^\infty(\mathbb{R})$. Then the wave equation (1) is globally well posed for initial data*

$$(u_0, u_1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}),$$

with solutions satisfying

$$u \in C(\mathbb{R}; H^1(\mathbb{R})) \cap C^1(\mathbb{R}; L^2(\mathbb{R})).$$

Inspired by the works of Lindblad and Tao [2] and Wei and Yang [3] on the one-dimensional wave equation with a non-localized defocusing nonlinearity, we next establish uniform L^∞ bounds for the solution, as well as decay on bounded spatial regions. This behavior contrasts with that of solutions to the linear wave equation, for which local decay may fail.

A novel aspect of our work is the study of scattering and the corresponding inverse scattering problem in the presence of a spatially localized nonlinearity. To state these results, let

$$\mathcal{X} = \left\{(u_0, u_1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}) : (u_0, u_1) \text{ are compactly supported}\right\},$$

and define the weighted energy space

$$\mathcal{Y} = \dot{H}^1(\mathbb{R}, \langle x \rangle^{-1}\, dx) \times L^2(\mathbb{R}, \langle x \rangle^{-1}\, dx), \qquad \langle x \rangle = (1 + |x|^2)^{1/2}.$$

Theorem 0.2 (–, Campos, Murphy (2026)) *Let K be admissible and let $(u_0, u_1) \in \mathcal{X}$. If u denotes the corresponding global solution to (1), then there exists a unique pair*

$$(U_+, V_+) \in \mathcal{Y}$$

such that

$$\lim_{t \to +\infty} \| S(-t)(u(t), \partial_t u(t)) - (U_+, V_+) \|_{\mathcal{Y}} = 0.$$

The preceding theorem allows us to define the scattering map

$$\Phi_K : \mathcal{X} \longrightarrow \mathcal{Y}, \qquad \Phi_K(u_0, u_1) = (U_+, V_+).$$

Our inverse scattering result shows that this map determines the localization coefficient K uniquely and, in fact, provides a quantitative stability estimate.

Theorem 0.3 (–, Campos, Murphy (2026)) *Let K_1 and K_2 be admissible, and let*

$$\Phi_1, \Phi_2 : \mathcal{X} \longrightarrow \mathcal{Y}$$

be the corresponding scattering maps. Then

$$\|K_2 - K_1\|_{L^\infty} \lesssim_{K_1,K_2} \left[\sup_{(u_0,u_1)\in\mathcal{X}\setminus\{0\}} \frac{\|\Phi_2(u_0,u_1) - \Phi_1(u_0,u_1)\|_{\mathcal{Y}}}{\|(u_0,u_1)\|_{\dot{H}^1\times L^2}} \right]^{1/8}.$$

In particular, if $\Phi_1 \equiv \Phi_2$, then $K_1 \equiv K_2$.

The first of these results shows that solutions to our model become asymptotically linear when measured in a suitable weighted energy space. Although the solutions need not decay globally in L^∞, they exhibit decay on every bounded spatial interval. The second result shows that the nonlinear scattering behavior contains enough information to recover the localization coefficient $K(x)$ uniquely. More precisely, the scattering map not only determines K, but does so in a quantitatively stable manner.

References

[1] Luccas Campos, Jason Murphy, and Renzo Scarpelli. Scattering and nonlinear recovery for the 1d nlw with localized nonlinearity, 2026.

[2] Hans Lindblad and Terence Tao. Asymptotic decay for a one-dimensional nonlinear wave equation. *Anal. PDE*, 5(2):411–422, 2012.

[3] Dongyi Wei and Shiwu Yang. Asymptotic decay for defocusing semilinear wave equations in $\mathbb{R}^{1+1}$. *Ann. PDE*, 7(1):Paper No. 9, 26, 2021.