# CHAOS AND SYMMETRY IN SKEW LOGISTIC COUPLINGS

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Abstract.

We investigate the skew logistic coupling as a quadratic planar endomorphism, focusing on the construction of optimal compact sets through the graph transform technique combined with a symmetry condition. By exploiting properties of unimodal functions together with symmetry, we show that every point in the interior of the compact set has a bounded orbit. This framework provides a rigorous foundation for numerical implementations and connects classical geometric methods with contemporary computational approaches. Our results extend the scope of invariant set constructions in non-invertible planar dynamics, offering new insights into the interplay between symmetry and boundedness.

## 1. Introduction

In [9], Anzai introduced an interesting class of systems, the so-called skew product transformations. These dynamical systems are defined as follows: given two topological spaces $X, Y$, we consider functions $f : X \to X$ y $g : X \times Y \to Y$ and form the endomorphism $F : X \times Y \to X \times Y$ as

$$F(x, y) := (f(x), g(x, y)).$$

These maps are called *skew product*. The space $X$ is called the *basis*, $Y$ is known as the *phase space* and $f$ is the *basis or driver dynamical system*. Since their introduction and in particular in recent years skew product transformations have been widely studied. In cite [9], they were studied to construct interesting and manageable examples in ergodic theory. In [6, 7] ergodic properties of skew products are studied, in particular, the Abramov-Roklhin formula is established as well as entropy calculations for skew products. In [36] they appear in the study of automorphisms of nilmanifolds. Other dynamic properties or applications can be found in [8, 13, 14, 17, 18, 23, 27, 28, 31, 37] and references therein.

A prominent class of skew product transformations arises when considering $g(x, y) = f(y) + \epsilon(x - y)$, that is, the skew maps $F(x, y) = (f(x), f(y) + \epsilon(x - y))$, these are called skew coupling. Such maps have been considered relevant models for the transmission of information through chaotic signals, and are also of interest for system synchronization, see for instance [20, 21]. A special case of these systems occurs when the driving system is the logistics map, $f_\mu(x) = \mu x(1 - x)$. In this case, the two dimensional system on two parameters

$$F_{\mu,\epsilon}(x, y) := (f_\mu(x), f_\mu(y) + \epsilon(x - y)),\ x, y \in \mathbb{R}, \tag{1}$$

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where $\mu > 1$ and $\epsilon \in (0, 1)$. This system is called the *logistic skew coupling*. Our goal in this work is to study different dynamic scenarios in the system (1) when the parameters $\mu, \epsilon$ vary.

Following the geometric tradition established by Palis and de Melo, invariant sets and homoclinic tangencies have played a central role in understanding the complexity of dynamical systems. Newhouse later revealed how persistent tangencies generate chaotic phenomena and an abundance of sinks, highlighting the delicate balance between stability and instability. Our work continues this line of inquiry by introducing a symmetry condition as a structural mechanism that ensures the existence and closure of invariant Jordan curves in skew logistic couplings. This perspective not only resonates with the classical framework but also provides a new geometric principle that coexists with chaos, enriching the theory of invariant sets.

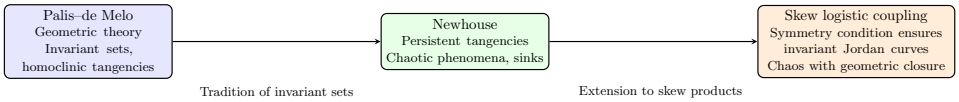



FIGURE 1. Conceptual transition: Palis–de Melo → Newhouse → Skew logistic coupling with symmetry condition

The historical development of the $\lambda$-lemma, introduced by Palis and Pugh in the 1970s, marked a turning point in the geometric theory of dynamical systems. This result revealed how transversal curves are stretched under iteration, approaching unstable manifolds in a characteristic shape reminiscent of the Greek letter $\lambda$. The lemma became a cornerstone in the study of homoclinic tangencies and chaotic dynamics, later expanded through singular versions and applications to bifurcation theory. In this classical framework, our contribution resonates with the same spirit: while the $\lambda$-lemma highlights deformation and convergence near invariant manifolds, we introduce a symmetry condition that ensures the existence and closure of invariant Jordan curves in skew logistic couplings. This perspective situates our work within the geometric tradition while extending it to a new structural principle in the context of skew products.

## 2. Preliminaries

We begin with some basic properties about the dynamics of (1). To simplify the computations, we define for each $x \in \mathbb{R}$

$$h_x(y) = (\mu - \epsilon)y - \mu y^2 + \epsilon x, \quad \text{with } y \in \mathbb{R}; \tag{2}$$

obviously $F_{\mu,\epsilon}(x, y) = (f_\mu(x), h_x(y))$, for all $x, y \in \mathbb{R}$. We emphasize that for every $n \geq 1$ and $x, y \in \mathbb{R}$ one has

$$F^n_{\mu,\epsilon}(x, y) = (f^n_\mu(x), h_{x,n}(y)),$$

where

$$h_{x,n}(y) = (h_{f^{n-1}_\mu(x)} \circ \cdots \circ h_{f_\mu(x)} \circ h_x)(y).$$

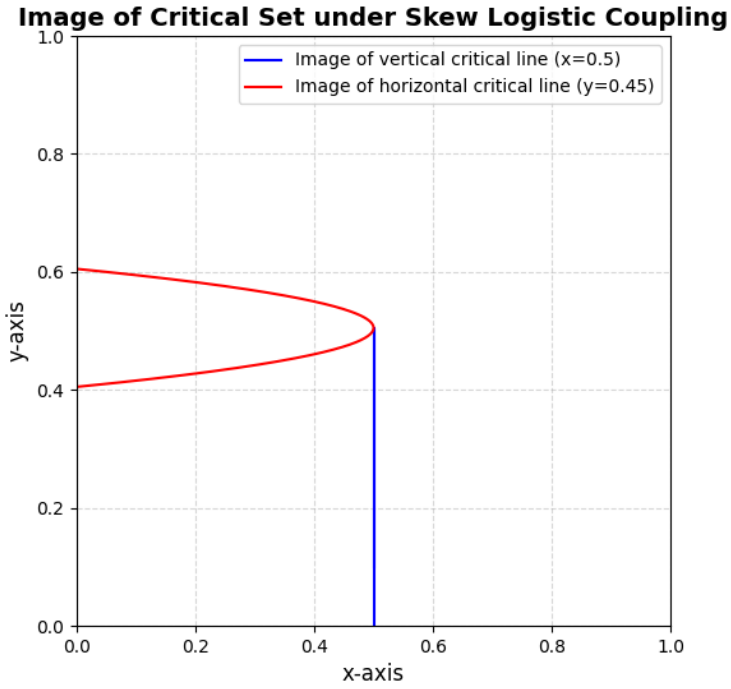



FIGURE 2. Critical image

2.1. **Conjugation between $h_x$ and the Logistic Map.** Clearly $h'_x(y) = \mu - \epsilon - 2\mu y$, so the unique critical point of $h_x$ is $d = \frac{1}{2}\left(1 - \frac{\epsilon}{\mu}\right)$ and the maximum value of $h_x$ is $M_x = \frac{(\mu-\epsilon)^2}{4\mu} + \epsilon x$. It is also clear that a point is fixed for $h_x$ if and only if it is a root of $(\mu - \epsilon - 1)y - \mu y^2 + \epsilon x = 0$. Therefore $h_x$ has fixed points if and only if $x > -\frac{(\mu-\epsilon-1)^2}{4\epsilon\mu}$; in that case, these fixed points are given by

$$\begin{cases} a(x) = \dfrac{1}{2\mu}\left(\mu - \epsilon - 1 - \sqrt{(\mu-\epsilon-1)^2 + 4\epsilon\mu x}\right) \\ \widehat{a}(x) = \dfrac{\mu - \epsilon - 1}{\mu} - a(x) \end{cases} \tag{3}$$

obviously $a(x)$ is the smaller root of $(\mu - \epsilon - 1)y - \mu y^2 + \epsilon x = 0$. It is also clear that if $x$ is a fixed point of $f_\mu$, then $(x, a(x))$ is the fixed point of $F_{\mu,\epsilon}$ in the fiber $\{x\} \times \mathbb{R}$ with the smallest ordinate; more precisely

$$a(0) = \begin{cases} 0, \text{ if } \mu > 1 + \epsilon \\ \frac{\mu-\epsilon-1}{\mu}, \text{ if } \mu \leq 1 + \epsilon \end{cases} \quad \text{and} \quad a\left(\tfrac{\mu-1}{\mu}\right) = -\tfrac{\epsilon}{\mu}.$$

Let $b(x)$ denote the other preimage of $a(x)$ under $h_x$; that is

$$b(x) = 1 - \frac{\epsilon}{\mu} - a(x), \tag{4}$$

in other words

$$b(x) = \frac{1}{2\mu}\left(\mu - \epsilon + 1 + \sqrt{(\mu - \epsilon - 1)^2 + 4\epsilon\mu x}\right); \tag{5}$$

clearly $a(x) + b(x) = 2d$, $a(x) \leq 0$ and $b(x) > 0$.

Fix $x \in [0, 1]$, we consider

$$\alpha(x) := \frac{1}{b(x) - a(x)}, \quad \beta(x) := \frac{-a(x)}{b(x) - a(x)}, \text{ and } \rho(x) := \frac{\mu}{\alpha(x)}. \tag{6}$$

Thus we can see that the map $h_x$ is conjugated to $f_{\rho(x)}$. More precisely, in [3] was established the following

**Lemma 2.1.** *For each $x \in [0, 1]$ and $\varphi : \mathbb{R} \to \mathbb{R}$ given by $\varphi(t) = \alpha(x)t + \beta(x)$, we have $f_{\rho(x)} = \varphi \circ h_x \circ \varphi^{-1}$, where $\alpha, \beta$, and $\rho$ are as in (6).*

Note that the intervals $I_x = [a(x), b(x)]$ are strictly increasing: $I_{x_1} \subsetneq I_{x_2}$ whenever $x_2 > x_1$. It is also clear that

$$\begin{cases} h_x(t) < t, \quad \text{for all } t < a(x), \\ h_x((b(x), +\infty)) = (-\infty, a(x)), \\ h_x^n(t) \to -\infty \quad \text{as } n \to +\infty, \text{ for every } t \notin [a(x), b(x)]. \end{cases} \tag{7}$$

Additionally, since $\{x\} \times \mathbb{R}$ is mapped by $F_{\mu,\epsilon}$ into $\{f_\mu(x)\} \times \mathbb{R}$ and $f_\mu^n(t) \to -\infty$ as $n \to +\infty$ for all $t \notin [0, 1]$, the bounded orbits of $F_{\mu,\epsilon}$ lie in the strip $[0, 1] \times \mathbb{R}$. This is the reason why only $x \in [0, 1]$ is considered when analyzing points with bounded orbits. Note further that if $x$ is any point in $[0, 1]$, then in the interval $I_x$ there are points whose positive $h_x$-orbit leaves $I_x$ if and only if $\mu > 4\alpha(x)$; this follows from the well-known property that in $[0, 1]$ there are points whose $f_\rho$-orbit leaves this interval if and only if $\rho > 4$. In other words: the maximum value $M_x$ of the function $h_x$ exceeds $b(x)$ if and only if $(b(x) - a(x))\mu > 4$. Since

$$b(x) - a(x) = \frac{1}{\mu}\left(1 + \sqrt{(\mu - \epsilon - 1)^2 + 4\epsilon\mu x}\right),$$

we have

$$M_x > b(x) \quad \text{if and only if} \quad (\mu - \epsilon - 1)^2 + 4\epsilon\mu x > 9; \tag{8}$$

in particular, $M_x > b(x)$ for every $x \in [0, 1]$ if and only if $\mu > 4 + \epsilon$.

**Remark 2.1.** *For more details on the location of the invariant compact set and the bifurcation diagram of the skew logistic coupling, see [3].*

2.2. **Detection of a Suitable Compact Set.** The compact set $Q$ that we describe below is an approximation to the optimal compact set with the property that its complement (a neighborhood of $\infty$) is positively invariant under $F_{\mu,\epsilon}$ and every point there has an unbounded positive orbit; that is, every point in $Q^c$ has an empty $\omega$-limit set.

First recall that for any fixed real number $\beta$, the image of the horizontal line $y = \beta$ is the parabola $P(\mu, \epsilon, \beta)$ given by the equation:

$$x = \tfrac{\mu}{4} - \tfrac{\mu}{\epsilon^2}\left(y - (\mu - \epsilon)\beta + \mu\beta^2 - \tfrac{\epsilon}{2}\right)^2. \tag{9}$$

Note that using the function $h_x$ ($h_x(y) = f_\mu(y) - \epsilon y + \epsilon x$), the above expression becomes:

$$x = \tfrac{\mu}{4} - \tfrac{\mu}{\epsilon^2}\left(y - h_0(\beta) - \tfrac{\epsilon}{2}\right)^2. \tag{10}$$

Obviously, when $\beta = \frac{1}{2}\left(1 - \frac{\epsilon}{\mu}\right)$ this parabola corresponds to the critical values:

$$x = \tfrac{\mu}{4} - \tfrac{\mu}{\epsilon^2}\left(y - \tfrac{\mu}{4} - \tfrac{\epsilon^2}{4\mu}\right)^2, \quad x \le \tfrac{\mu}{4}, \tag{11}$$

whose lower and upper branches are given, respectively, by

$$y = \tfrac{\mu^2+\epsilon^2}{4\mu} - \tfrac{\epsilon}{2}\sqrt{1 - \tfrac{4x}{\mu}}, \quad y = \tfrac{\mu^2+\epsilon^2}{4\mu} + \tfrac{\epsilon}{2}\sqrt{1 - \tfrac{4x}{\mu}}. \tag{12}$$

In general, the upper and lower branches of the parabola $P(\mu, \epsilon, \beta)$ are given by

$$y = h_0(\beta) + \tfrac{\epsilon}{2} \pm \tfrac{\epsilon}{2}\sqrt{1 - \tfrac{4x}{\mu}}, \quad x \le \tfrac{\mu}{4}; \tag{13}$$

in particular, the values of $y$ in the above expression when $x = 0$ are

$$y = h_0(\beta) + \tfrac{\epsilon}{2} \pm \tfrac{\epsilon}{2};$$

that is, $y = h_0(\beta)$ and $y = h_1(\beta)$. We seek a value of $\beta$ such that the value of $y$ in the upper branch of the parabola given by (13) is exactly $\beta$. Such a value of $\beta$ corresponds to the fixed points of the function $h_1$: $a(1)$ and $\widehat{a}(1)$. We take the smaller of these, $a(1)$, and define the compact set

$$Q = [0, 1] \times [a(1), b(1)].$$

From the discussion above, the following properties hold:

(1) $Q \cap P(a(1)) = Q \cap P(b(1)) = \{(0, a(1))\}$.
(2) Given $\beta \notin [a(1), b(1)]$, if $(0, y_\beta)$ is the intersection point between $P(\epsilon, \mu, \beta)$ and $Q$ (in fact the upper branch of $P(\epsilon, \mu, \beta)$), then $y_\beta < a(1)$. Moreover, if $\beta < a(1)$, then $y_\beta < \beta$; hence the ordinate of $F_{\mu,\epsilon}(x, \beta)$, for any $x$, is less than $\beta$.

Define

$$Q^+ = ([0, 1] \times \mathbb{R}) \cap \{(x, y) : y > b(1)\}, \quad Q^- = ([0, 1] \times \mathbb{R}) \cap \{(x, y) : y < a(1)\}.$$

From the above it follows that $F_{\mu,\epsilon}(Q^+) \subset Q^-$ and $F_{\mu,\epsilon}(Q^-) \subset Q^-$. In fact, this also implies that points in $Q^-$ have unbounded orbits. This is actually a particular case of the general property stated in the following result (see [38], Lemma 2.1).

**Lemma 2.2.** *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuous, and let $W \subset \mathbb{R}^n$ be such that $F(W) \subset W$. If there exists a continuous function $L : W \to \mathbb{R}$ such that the orbital difference $\Delta L = L \circ F - L$ is definite (positive or negative), then the points in $W$ have unbounded orbits.*

A function as described in the statement above is called a *Lyapunov function* associated with $F$ on the set $W$. Note that in our case it suffices to consider $W = Q^-$ and $L : Q^- \to \mathbb{R}$ defined by $L(x, y) = y$ for each $(x, y) \in Q^-$. From this we obtain that $\infty$ is an attractor for $F_{\mu,\epsilon}$; moreover, if $ext\, Q$ is the unbounded component of the complement of $Q$ ($int\, Q$ is the respective bounded component), then $ext\, Q \subset B_\infty$ and thus we have

$$B_\infty = \bigcup_{n\ge 0} F_{\mu,\epsilon}^{-n}(ext\, Q), \quad \Lambda := \mathbb{R}^2 \setminus B_\infty = \bigcap_{n\ge 0} F_{\mu,\epsilon}^{-n}(Q). \tag{14}$$

Note that $\Lambda$ is the set of points whose orbits are bounded; also observe that $\Lambda$ is completely invariant; that is, $F_{\mu,\epsilon}^{-1}(\Lambda) = \Lambda$.

## 3. Hyperbolicity and chaos of the invariant compact set

In the study of dynamical systems, hyperbolicity is fundamental for understanding long-term stability as also chaotic behavior. We are interested in providing sufficient conditions for the hyperbolicity of the invariant set $Q$. Let a matrix be given $A \in M_{2x2}(\mathbb{R})$, we recall the spectral norm of this matrix defined as:

$$||A|| := max\{|\lambda| : Ax = \lambda x\}, \text{ with } x \in \mathbb{R}^2.$$

This norm is the largest eigenvalue of the symmetric matrix $A^tA$.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ an endomorphism of $C^1$, and let $K \subset \mathbb{R}^2$ a compact invariant set. The following result, well known in the literature, gives equivalent conditions for an invariant set to be a hyperbolic repelling set:

**Lemma 3.1.** *The following are equivalent*

*(1)* ***uniform expansion:*** *There exist constants $C > 0$, $\lambda > 1$ such that for every $p \in K$, all $v \in T_pM$ and all $n \geq 0$,*

$$\|Df_p^n(v)\| \geq C\lambda^n\|v\|.$$

*(2)* ***Operator norm greater than one:*** *There exists $\lambda > 1$ such that*

$$\|Df_p\| \geq \lambda, \quad \forall p \in K.$$

Using this result we can establish

**Proposition 3.1.** *If $\mu > 5 + \epsilon$ the invariant compact set $Q$ is a hyperbolic repelling set.*

*Proof.* The Jacobian of the skew logistic coupling can be expressed as

$$DF_{\mu,\epsilon}(x, y) = \begin{pmatrix} \mu - 2\mu x & 0 \\ \epsilon & \mu - 2\mu y - \epsilon \end{pmatrix},$$

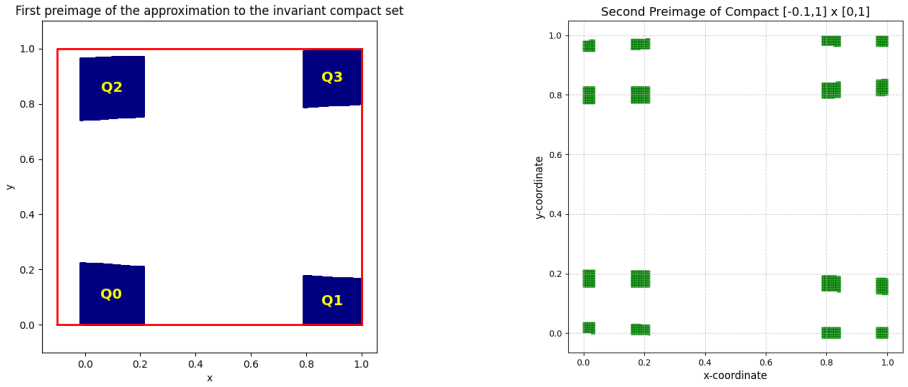
a direct account gives us $f'_\mu(x) > 1$, for all $x \in [0, 1]$ with $\mu > 2 + \sqrt{5}$, thus $||DF_{\mu,\epsilon}(x, y)|| > 1$. Therefore, we get

$$||DF_{\mu,\epsilon}^n(x, y)|| \geq C\lambda^n,$$

for $C > 0$ and $\lambda > 1$. This concludes the result.

□

Based on hyperbolicity, we seek to go deeper into the dynamic complexity of the invariant set, focusing our attention on the case $\mu > 5 + \epsilon$. To understand this behavior, let us consider the first preimage $F_{\mu,\epsilon}^{-1}(Q)$ a straightforward calculation tells us that it is made up of four disjoint compact sets each of them with a smaller area than that of $Q$, we will denote these by $Q_0, Q_1, Q_2$ and $Q_3$. Thus, for every $n \geq 1$, $F_{\mu,\epsilon}^{-n}(Q)$ it is the union of $4^n$ disjoint compact sets. We shall denote these compact sets accordingly. Iterating this process, by removing open regions and retaining only disjoint subsets. This construction yields a sequence of compact subsets with decreasing area, which in turn, yields a Cantor set in the plane, that is, a compact, perfect, and totally disconnected set. We denote by $\Lambda_{\mu,\epsilon} = \bigcap_{n\geq 0} F_{\mu,\epsilon}^{-n}(Q)$ to this set. As is common in symbolic dynamics, let $\Sigma_4 := \{0, 1, 2, 3\}^{\mathbb{N}\cup\{0\}}$ be the




(A) First preimage of the approximation to the invariant compact set

(B) Second preimage of the approximation to the invariant set

FIGURE 3. Illustration of the formation of the Cantor set.

space of infinite sequences on four symbols with the usual metric $d_2$. Also we consider the shift map $\sigma : \Sigma_4 \to \Sigma_4$.

Doing

$$Q_{i_0 i_1 i_2 \ldots i_n} := \{x \in Q : x \in Q_{i_0}, F_{\mu,\epsilon}(x) \in Q_{i_1}, ..., F^n_{\mu,\epsilon}(x) \in Q_{i_n}\},$$

hence $F_{\mu,\epsilon}(Q_{i_0 i_1 i_2 \ldots i_n}) = Q_{i_1 i_2 i_3 \ldots i_n}$ or $z \in Q_{i_0 i_1 i_2 \ldots i_n} \leftrightarrow F^j_{\mu,\epsilon}(z) \in Q_{i_j}$.

Thus we can define the function of the itinerary of the orbits $\phi : \Lambda_{\mu,\epsilon} \to \Sigma_4$, more precisely for $z \in \Lambda_{\mu,\epsilon}$ we get $\phi(z) := (i_0, i_1, i_2, ..., i_n, \ldots)$.

It is clear that this function is surjective and continuous. Therefore we have

**Lemma 3.2.** *The map $\phi$ is a semiconjugacy between $(\Lambda_{\mu,\epsilon}, F_{\mu,\epsilon})$ and $(\Sigma_4, \sigma)$, that is, the following diagram is commutative:*

$$\begin{array}{ccc} \Lambda_{\mu,\epsilon} & \xrightarrow{F_{\mu,\epsilon}} & \Lambda_{\mu,\epsilon} \\ \phi \downarrow & & \downarrow \phi \\ \Sigma_4 & \xrightarrow[\sigma]{} & \Sigma_4 \end{array}$$

*Therefore*

$$\phi \circ F_{\mu,\epsilon} = \sigma \circ \phi.$$

Using classical arguments based on the hyperbolicity of $\Lambda_{\mu,\epsilon}$ we get that $\phi$ is injective.

Consequently, combining the previous lemma and the proposition 3.1 we have the following

**Theorem 3.3.** *If $\mu > 5 + \epsilon$, then $(\Lambda_{\mu,\epsilon}, F_{\mu,\epsilon})$ is topologically conjugate to the one–sided shift on four symbols.*

As an immediate consequence we have

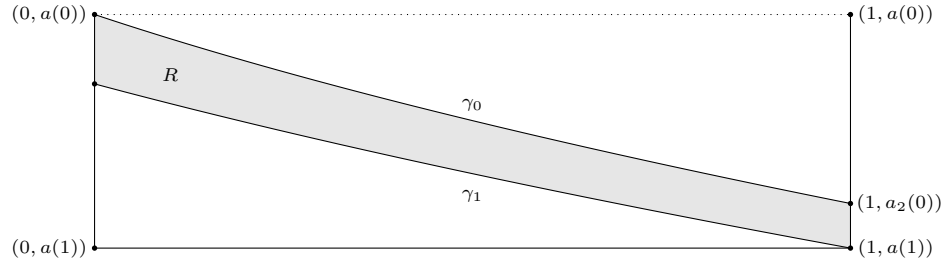
**Corollary 3.4.** *The system $(\Lambda_{\mu,\epsilon}, F_{\mu,\epsilon})$ is chaotic in the sense of Devaney:*

## 4. Invariant curves

In this section, we construct invariant Jordan curves associated with the skew logistic coupling. These curves arise as the immediate boundary of the basin of attraction of infinity. Using the graph transform method, we prove the existence of a compact invariant curve $\varphi_{\mu,\epsilon}$.

This curve separates bounded dynamics from divergent trajectories and provides a geometric framework for understanding the chaotic behavior of the system. Moreover, the invariant curve plays a crucial role in the classification of attractors and in the description of the topological structure of the phase space. Let us consider the rectangle $Q = [0,1] \times [a(1), b(1)]$, where $a(x)$ y $b(x)$ are as defined in equation (3) and (4) ; in fact, we shall denote in this manuscript ; for example, for each real number $v$, $P_v$ it is the parabola which is the preimage of the horizontal line $y = v$, see Figure (4). Suppose that $\mu \leq 4$ and $\epsilon$ is any in the interval $(0, 1)$.

Let $\gamma_0$ and $\gamma_1$ the lower arcs of the parabolas $P_{a(0)}$ y $P_{a(1)}$ in the interval $[0, 1]$; and let $\gamma_0^+$ , $\gamma_1^+$ the upper arcs of the same parabolas in the same interval. Note that the endpoints on the left of $\gamma_0$ y $\gamma_0^+$are respectively: $(0, a(0))$ and $(0, b(0))$ while the endpoints on the right of $\gamma_1$ and $\gamma_1^+$ are $(1, a(1))$ y $(1, b(1))$; rememberv that $a(0)$ and $b(0)$ are the solutions of $h_0(t) = a(0)$, and $a(1)$, $b(1)$are the values that solve $h_1(t) = a(1)$. The remaining endpoints of the upper arcs can also be described easily; in fact, let us denote by $(0, a_1(0)), (1, a_2(0))$ y $(1, a_3(0))$the other preimages of the fixed point $(0, a(0))$,It is obvious that $a_1(0) = b(0)$ y $a_2(0) \neq a_3(0)$,We shall suppose $a_2(0) < a_3(0)$;then the endpoints on the right of $\gamma_0$ and $\gamma_0^+$ are respectively $(1, a_2(0))$ y $(1, a_3(0))$.The endpoints of $\gamma_1$ and $\gamma_1^+$on the vertical $x = 0$,these are the preimages of $(0, a(1))$ on this verticall.




In particular, if $\varphi_1$ is the function that defines the curve described above, then the unbounded component $ext\,\Gamma_{\varphi_1}$ of the complement of $\Gamma_{\varphi_1}$ is contained in $B_\infty$;

. Consequently, the closure of the corresponding bounded component $cl(int\,\Gamma_{\varphi_1})$, is a compact subset of $Q$ containing the points with bounded orbits. Although this compact set is contained in $Q$, it is not the minimal one. One of our objectives is precisely to determine the smallest of such compact sets. For this purpose, the following lemma—despite its very simple proof—is fundamental to our aims.

In particular if $\varphi_1$ s the function that defines the curve $\gamma_1$ described above, then the unbounded component $ext\,\Gamma_{\varphi_1}$ of the complement of $\Gamma_{\varphi_1}$ is contained in $B_\infty$; Hence, the closure of the corresponding bounded component, $cl(int\,\Gamma_{\varphi_1})$ is a compact subset of $Q$ containing the points with bounded orbit. Although this compact set is contained in $Q$,, it is not the minimal one. One of our objectives is precisely to determine the minimal such compact set; for this purpose, the following lemma,

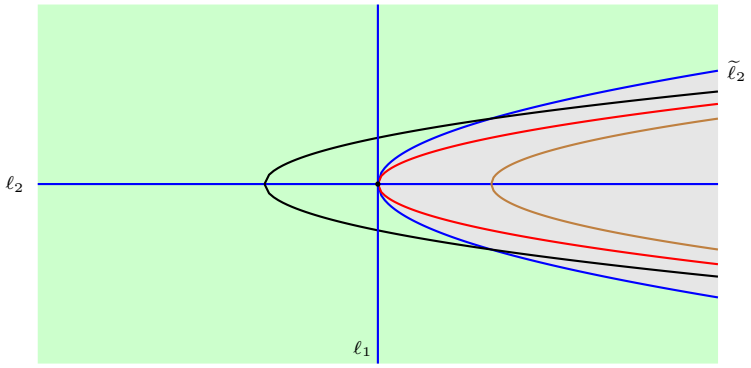



FIGURE 4. Location of the preimage of a horizontal line. It is distinguished by colors (brown, red, black $v > \frac{\mu^2+\epsilon^2}{4\mu}$, $v = \frac{\mu^2+\epsilon^2}{4\mu}$ y $v < \frac{\mu^2+\epsilon^2}{4\mu}$. The smaller the value of $v$, the vertex of the parabola lies further to the left $P_v$.

although very simple to prove, is fundamental to our aims. Let us denote by $R$ the region bounded by the curves $\gamma_0$ and $\gamma_1$. In view of the way the parabolas In view of the way the parabolas $P_v$, are distributed, we can say that for any $a(1) \leq v \leq a(0)$ , the arc of the parabola $P_v$in the interval $[0,1]$ lies between the curves $\gamma_0$ and $\gamma_1$,since obviously $a(0) < \min h_0$; it follows that for every $(u,v) \in R$ with $u \leq \frac{\mu}{4}$, the two points $(x_-(u), y_-^-(u,v))$ y $(x_+(u), y_-^+(u,v))$ are well defined and, moreover, they lie in $R$; remember that

$$
\begin{aligned}
x_-(u) &= \tfrac{1}{2}\left(1 - \sqrt{1 - \tfrac{4u}{\mu}}\right),\ x_+(u) = \tfrac{1}{2}\left(1 + \sqrt{1 - \tfrac{4u}{\mu}}\right) \\
y_-^-(u,v) &= \tfrac{1}{2\mu}\left(\mu - \epsilon - \sqrt{(\mu-\epsilon)^2 + 4\epsilon\mu x_-(u) - 4\mu v}\right) \\
y_+^-(u,v) &= \tfrac{1}{2\mu}\left(\mu - \epsilon + \sqrt{(\mu-\epsilon)^2 + 4\epsilon\mu x_-(u) - 4\mu v}\right) \\
y_-^+(u,v) &= \tfrac{1}{2\mu}\left(\mu - \epsilon - \sqrt{(\mu-\epsilon)^2 + 4\epsilon\mu x_+(u) - 4\mu v}\right) \\
y_+^+(u,v) &= \tfrac{1}{2\mu}\left(\mu - \epsilon + \sqrt{(\mu-\epsilon)^2 + 4\epsilon\mu x_+(u) - 4\mu v}\right)
\end{aligned}
\tag{15}
$$

Note that the functions $x_- : [0, \frac{\mu}{4}] \to [0, \frac{1}{2}]$ and $x_+ : [0, \frac{\mu}{4}] \to [\frac{1}{2}, 1]$ Note that the functions are strictly increasing or decreasing, depending on the domain under consideration., $x_-(u) + x_+(u) = 1$ para cada $u \in [0, \frac{\mu}{4}]$ Moreover, their inverses are precisely the restrictions of the original functions to the corresponding domains. This property guarantees that the monotonic behavior of is preserved under inversion, ensuring a consistent framework for analyzing the associated invariant curves and their complementary components, $f_\mu$ on the intervals $[0, \frac{1}{2}]$ y $[\frac{1}{2}, 1]$; traversed in a one-to-one manner, establishing a bijective correspondence between the function and its inverse $\{x_-(t) : t \in [0, \frac{\mu}{4}]\} \cup \{x_+(t) : t \in [0, \frac{\mu}{4}]\}$ "traverses in a one-to-one manner" or more formally, establishes a bijective correspondence with $[0,1]$.

For any fixed $v \in \mathbb{R}$, the horizontal line $L_v$ given by $y = v$ always intersects $cl(Z_2 \cup Z_4)$; luego $F_{\mu,\epsilon}^{-1}(L_v) = P_v$.We distinguish three mutually exclusive cases:

$$v < \tfrac{\mu^2+\epsilon^2}{4\mu},\ v = \tfrac{\mu^2+\epsilon^2}{4\mu} \text{ y } v > \tfrac{\mu^2+\epsilon^2}{4\mu}.$$

The following two figures show the position of the lines $L_v$ and their preimage according to the cases mentioned above.

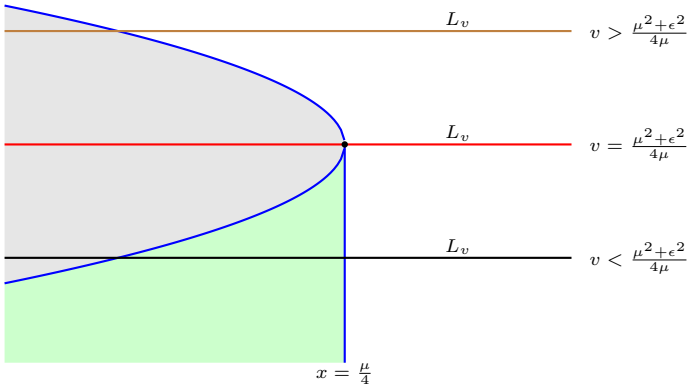



FIGURE 5. The intersection of any horizontal line with $cl(Z_2 \cup Z_4)$ is a ray extending to the left.

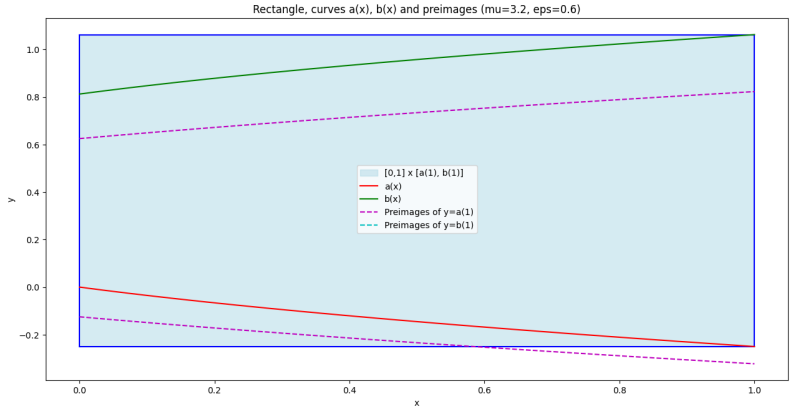



FIGURE 6. Preimages of $y = a(1)$ y $y = b(1)$.

**Lemma 4.1.** *Given $\mu \in (1,4]$ and $\epsilon \in (0,1)$, if $\varphi : [0,1] \to \mathbb{R}$ is any function whose graph$(\varphi)$ is contained in $R$, then the set*

$$\{(x_-(t), y_-^-(t,\varphi(t))) : t \in [0, \tfrac{\mu}{4}]\} \cup \{(x_+(t), y_-^+(t,\varphi(t))) : t \in [0, \tfrac{\mu}{4}]\} \tag{16}$$

*define the graph of a function $\Phi(\varphi) : [0,1] \to \mathbb{R}$.*

*Proof.* Since the points $(x_-(t), y_-^-(t,\varphi(t)))$ y $(x_+(t), y_-^+(t,\varphi(t)))$ are defined for all $t \in [0, \frac{\mu}{4}]$, it suffices to observe that

$$\Phi(\varphi)(s)) = \begin{cases} y_-^-(f_\mu(s), \varphi(f_\mu(s))), & \text{si } s \in [0, \frac{1}{2}] \\ y_-^+(f_\mu(s), \varphi(f_\mu(s))), & \text{si } s \in [\frac{1}{2}, 1] \end{cases} ; \tag{17}$$

define a function from $[0,1]$ en $\mathbb{R}$ whose graph $graph(\Phi(\varphi))$ is the set given by (16). Crearly of (15) it follows that for each $\varphi \in \mathcal{C}$ and each $s \in [0,1]$, agreed

$$\Phi(\varphi)(s) = \tfrac{1}{2\mu}\left(\mu - \epsilon - \sqrt{(\mu-\epsilon)^2 + 4\epsilon\mu s - 4\mu\varphi(f_\mu(s))}\right). \tag{18}$$

Note also that for each $s \in [0, \frac{1}{2}]$ and $t \in [0, \frac{\mu}{4}]$ with $s = x_-(t)$ we have

$$F_{\mu,\epsilon}(s, \Phi(\varphi)(s)) = (t, \varphi(t)) = F_{\mu,\epsilon}(1-s, \Phi(\varphi)(1-s)); \tag{19}$$

this latter is the **Symmetry condition**, and from the above:

$$F_{\mu,\epsilon}(graph(\Phi(\varphi))) \subset graph(\varphi)$$

.

□

We denote by $\mathcal{C}$ to the set of all continuous functions $\varphi : [0,1] \to \mathbb{R}$ tales que $graph(\varphi) \subset R$ y $F_{\mu,\epsilon}(t, \varphi(t)) = F_{\mu,\epsilon}(1-t, \varphi(1-t))$ for every $t \in [0,1]$. From these conditions, it is straightforward to verify that $\varphi \in \mathcal{C}$, them

$$\varphi(1-t) = \tfrac{1}{2\mu}\left(\mu - \epsilon - \sqrt{(2\mu\varphi(t) - \mu + \epsilon)^2 - 4\epsilon\mu(2t-1)}\right) \text{ for all } t \in [0,1]; \tag{20}$$

Note also that for every $t \in [0,1]$ y $\varphi \in \mathcal{C}$ it holds

$$\varphi(t) \leq \tfrac{1}{2\mu}\left(\mu - \epsilon - \sqrt{(\mu-\epsilon)^2 + 4\epsilon\mu t - 4\mu a(0)}\right), \tag{21}$$

Recall that the expression in the upper right defines the function $\varphi_0$ which determines the curve $\gamma_0$ used in the definition of the region $R$.

On the other hand, observe that $\mathcal{C}$ endowed with the distance

$$d(\varphi, \psi) = \max_{s \in [0,1]} |\varphi(s) - \psi(s)|$$

is a complete metric space and y $\Phi$ given by (17)is an operator on $\mathcal{C}$.

For all $\varphi \in \mathcal{C}$, we denoted by $\Gamma_\varphi$ the Jordan curve given by the juxtaposition of:

$$\begin{aligned} &\Gamma^b_\varphi = graph(\varphi),\ \Gamma^t_\varphi = \{(s, \tfrac{\mu-\epsilon}{\mu} - \varphi(s)) : s \in [0,1]\}, \\ &\Gamma^l_\varphi = \{0\} \times [\varphi(0), \tfrac{\mu-\epsilon}{\mu} - \varphi(0)],\ \Gamma^r_\varphi = \{1\} \times [\varphi(1), \tfrac{\mu-\epsilon}{\mu} - \varphi(1)]. \end{aligned} \tag{22}$$

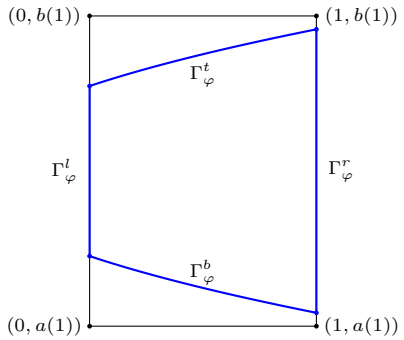



FIGURE 7. Illustration of the Jordan curve obtained from a function $\varphi$ en $\mathcal{C}$.

In particular, if $\varphi_1$ is the function that defines the curve $\gamma_1$ described above, then the unbounded component $ext\,\Gamma_{\varphi_1}$ of the complement of $\Gamma_{\varphi_1}$ is contained in $B_\infty$; consequently, the closure of the corresponding bounded component, $cl(int\,\Gamma_{\varphi_1})$, is a compact subset of $Q$ containing the points with bounded orbit. Although this compact set is contained in $Q$, it is not the minimal one. One of our objectives is precisely to determine the minimal such compact set; for this purpose, the following lemma, although its proof is very simple, is fundamental to our aims.

**Lemma 4.2.** *For every $1 < \mu \leq 4$ and $\epsilon \in (0,1)$, the operator $\Phi$ is a contraction on $(\mathcal{C}, d)$; that is, there exists a constant $0 < \lambda < 1$ such that $d(\Phi(\varphi), \Phi(\psi)) \leq \lambda d(\varphi, \psi)$, for all $\varphi, \psi \in \mathcal{C}$.*

*Proof.* Given $\varphi, \psi \in \mathcal{C}$, for all $s \in [0,1]$ it is true

(23)
$$\Phi(\varphi)(s) - \Phi(\psi)(s) = \begin{cases} y_-^-(f_\mu(s), \varphi(f_\mu(s))) - y_-^-(f_\mu(s), \psi(f_\mu(s))), & \text{si } s \in [0, \frac{1}{2}] \\ y_-^+(f_\mu(s), \varphi(f_\mu(s))) - y_-^+(f_\mu(s), \psi(f_\mu(s))), & \text{si } s \in [\frac{1}{2}, 1] \end{cases}.$$

Since $\varphi(s), \psi(s) \leq a(0) \leq 0$, from the definitions of $y_-^-$ y $y_-^+$, ver (15), it follows that
$$|\Phi(\varphi)(s) - \Phi(\psi)(s)| \leq \frac{d(\varphi, \psi)}{\sqrt{(\mu - \epsilon)^2 - 4\mu a(0)}}.$$
Now, since $a(0) = 0$ si $\mu \geq 1 + \epsilon$ y $a(0) = \frac{\mu - \epsilon - 1}{\mu}$ when $\mu < 1 + \epsilon$,we obtain
$$d(\Phi(\varphi), \Phi(\psi)) \leq \begin{cases} \dfrac{1}{\mu - \epsilon} d(\varphi, \psi), & \text{si } \mu \geq 1 + \epsilon \\ \dfrac{1}{2 + \epsilon - \mu} d(\varphi, \psi), & \text{si } \mu < 1 + \epsilon; \end{cases}$$

Thus, the proof of the lemma is complete.

□

As an immediate consequence of this simple lemma, one has the well-known Banach contraction theorem Banach (ver [1]), for all $\mu \in (0, 4]$ and $\epsilon \in (0, 1)$ there exists a function, hereafter denoted by $\varphi_{\mu,\epsilon} \in \mathcal{C}$ (point fixed de $\Phi$), such that $d(\Phi^n(\varphi), \varphi_{\mu,\epsilon}) \to 0$ when $n \to +\infty$, for all $\varphi \in \mathcal{C}$; this is, $\varphi_{\mu,\epsilon}$ is the global attractor of $\Phi$ en $\mathcal{C}$, remember that $\Phi^n = \Phi \circ \Phi^{n-1}$ for all $n \geq 1$.

With the above notations, the following statements are easily verified:

a) $ext\,\Gamma_{\varphi_{\mu,\epsilon}} \subset B_\infty$, wherw $\Gamma_{\varphi_{\mu,\epsilon}}$ is juxtaposition defined in (22).

b) The ends of $\Gamma^b_{\varphi_{\mu,\epsilon}}$ and $\Gamma^t_{\varphi_{\mu,\epsilon}}$ constitute the preimage $F^{-1}_{\mu,\epsilon}(0, a(0))$, obviously $(0, a(0))$ is the left endpoint of $\Gamma^b_{\varphi_{\mu,\epsilon}}$ and $(0, b(0))$ it is for $\Gamma^t_{\varphi_{\mu,\epsilon}}$.

c) The curve $\Gamma^b_{\varphi_{\mu,\epsilon}}$ It is positively invariant, that is $F_{\mu,\epsilon}(\Gamma^b_{\varphi_{\mu,\epsilon}}) \subset \Gamma^b_{\varphi_{\mu,\epsilon}}$; in fact, for each $x \in [0,1]$ we have $F_{\mu,\epsilon}(x, \varphi_{\mu,\epsilon}(x)) = (f_\mu(x), \varphi_{\mu,\epsilon}(f_\mu(x)))$, of where

$$\varphi_{\mu,\epsilon}(f_\mu(x)) = h_x(\varphi_{\mu,\epsilon}(x)), \text{ para todo } x \in [0,1]. \tag{24}$$

Note that $F_{\mu,\epsilon}(\Gamma^b_{\varphi_{\mu,\epsilon}}) = F_{\mu,\epsilon}(\Gamma^t_{\varphi_{\mu,\epsilon}}) = \{(t, \varphi_{\mu,\epsilon}(t)) : t \in [0, \frac{\mu}{4}]\}$ and $F^{-1}_{\mu,\epsilon}(\Gamma^b_{\varphi_{\mu,\epsilon}}) = \Gamma^b_{\varphi_{\mu,\epsilon}} \cup \Gamma^t_{\varphi_{\mu,\epsilon}}$; on the other hand, the curve $\Gamma^t_{\varphi_{\mu,\epsilon}}$ may or may not have a preimage.

d) The dynamics of $F_{\mu,\epsilon}$ restricted to $\Gamma^b_{\varphi_{\mu,\epsilon}}$ coincides with that of the logistic map $f_\mu$ on the interval $[0,1]$. Indeed, consider the map $\pi : \Gamma^b_{\varphi_{\mu,\epsilon}} \to [0,1]$ defined by $\pi(t, \varphi_{\mu,\epsilon}(t)) = t$, for each $t \in [0,1]$, This is a homeomorphism when the metric $d$

is restricted to a $\Gamma^b_{\varphi_{\mu,\epsilon}}$ and clearly $f_\mu(t) = (\pi^{-1} \circ F_{\mu,\epsilon} \circ \pi)(t)$ for each $t \in [0,1]$. In particular, this implies that the fixed point $(\frac{\mu-1}{\mu}, -\frac{\epsilon}{\mu})$ of $F_{\mu,\epsilon}$ is in $\Gamma^b_{\varphi_{\mu,\epsilon}}$.

e) The curve $\Gamma^l_{\varphi_{\mu,\epsilon}}$ It always has a preimage, not so in this case $\Gamma^r_{\varphi_{\mu,\epsilon}}$; in fact, the latter has no preimage if and only $\mu < 4$.

f) The curve $\Gamma^t_{\varphi_{\mu,\epsilon}}$ it is the graph of the function $\widetilde{\varphi}_{\mu,\epsilon} : [0,1] \to \mathbb{R}$ given for

$$\widetilde{\varphi}_{\mu,\epsilon}(x) = \tfrac{\mu-\epsilon}{\mu} - \varphi_{\mu,\epsilon}(x) \text{ whichever it may be } x \in [0,1]; \tag{25}$$

observe that $\widetilde{\varphi}_{\mu,\epsilon}(x) \geq b(0)$ for each $x \in [0,1]$.

### 4.1. **Immediate basin of infinity.**

**Theorem 4.3.** *For any $\epsilon \in (0,1)$ and $\mu \in (1,+\infty)$, if $b(0) \geq \max h_1$, then the Jordan curve $\Gamma_{\varphi_{\mu,\epsilon}}$ is the boundary of $B_\infty$; in particular, $\operatorname{ext}\Gamma_{\varphi_{\mu,\epsilon}} = B_\infty$, and hence every point in $\operatorname{cl}(\operatorname{int}\Gamma_{\varphi_{\mu,\epsilon}})$ has a bounded orbit.*

The proof of this result relies on an elementary property of unimodal functions; these are continuous functions $f : \mathbb{R} \to \mathbb{R}$ for which there exists $c \in \mathbb{R}$ such that $f$ is strictly increasing on $(-\infty, c]$ and strictly decreasing on $[c, +\infty)$. It is well known that if $[a,b]$ is an interval and $c \in (a,b)$, then

$$f([a,b]) = [\min\{f(a), f(b)\}, f(c)].$$

*Proof.* The first point to consider is the relative position of the set of critical values $J_1$ and the curve $\Gamma_{\varphi_{\mu,\epsilon}}$. Since $b(0) \geq \max h_1$ and $\Gamma^b_{\varphi_{\mu,\epsilon}}$ (except at $(0, a(0))$) lies below the horizontal line $y = a(0)$, it follows that the piece of the critical parabola $P(\mu, \epsilon, \frac{1}{2}(1 - \frac{\epsilon}{\mu}))$ lies below $\Gamma^t_{\varphi_{\mu,\epsilon}}$; see Figure (8). In this case, $\Gamma^t_{\varphi_{\mu,\epsilon}}$ admits no preimage, and $F^{-1}_{\mu,\epsilon}(\Gamma^l_{\varphi_{\mu,\epsilon}}) = \Gamma^l_{\varphi_{\mu,\epsilon}} \cup \Gamma^r_{\varphi_{\mu,\epsilon}}$. Hence, $\Gamma_{\varphi_{\mu,\epsilon}}$ is completely invariant:

$$F^{-1}_{\mu,\epsilon}(\Gamma_{\varphi_{\mu,\epsilon}}) = \Gamma_{\varphi_{\mu,\epsilon}},$$

which implies that every point on $\Gamma_{\varphi_{\mu,\epsilon}}$ has a bounded orbit. Now take any point $(x,y) \in \operatorname{int}\Gamma_{\varphi_{\mu,\epsilon}}$. From the above, it is clear that $h_{x,n}(y) < b(0)$ for each $n \geq 1$; recall that

$$h_{x,n} = h_{f^{n-1}_\mu(x)} \circ \cdots \circ h_x, \quad \text{and} \quad F^n_{\mu,\epsilon}(x,y) = (f^n_\mu(x), h_{x,n}(y)).$$

It follows that if the orbit of $(x,y)$ were unbounded, since $f^n_\mu(x) \in [0,1]$ for each $n \geq 0$, then $h_{x,n}(y) \to -\infty$ as $n \to +\infty$. However, from (24) and the aforementioned property of unimodal functions, one obtains by induction that

$$h_{x,n}(y) \geq \varphi_{\mu,\epsilon}(f^n_\mu(x)) \quad \text{for each } n \geq 1,$$

which implies a lower bound for the sequence $(h_{x,n}(y))_{n\geq 1}$, since $\varphi_{\mu,\epsilon}(f^n_\mu(x)) \geq a(0)$ for each $n \geq 0$ and all $x \in [0,1]$. □

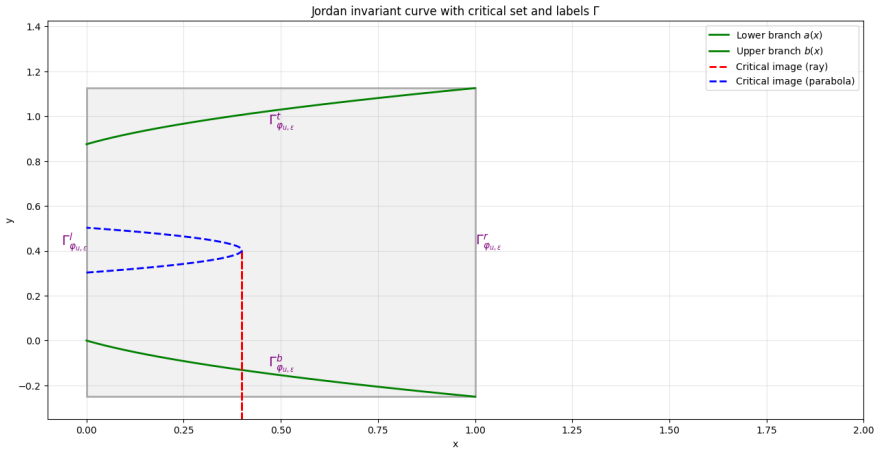



FIGURE 8. Approximation to the Jordan curve, $\mu = 1.6$ with its critical set.

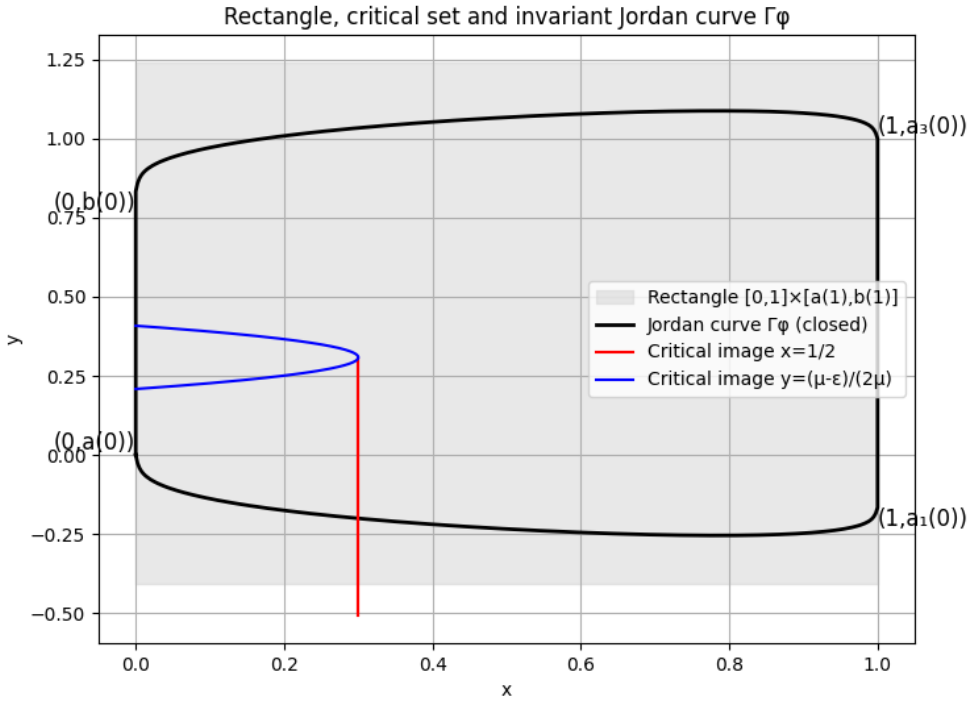



FIGURE 9. Illustration of a numerical approximation of the curve $\Gamma_{\varphi_\mu}$, the rectangle $Q$, and the set of critical values $J_1$ when $\mu = 1.2$ and $\varepsilon = 0.4$.

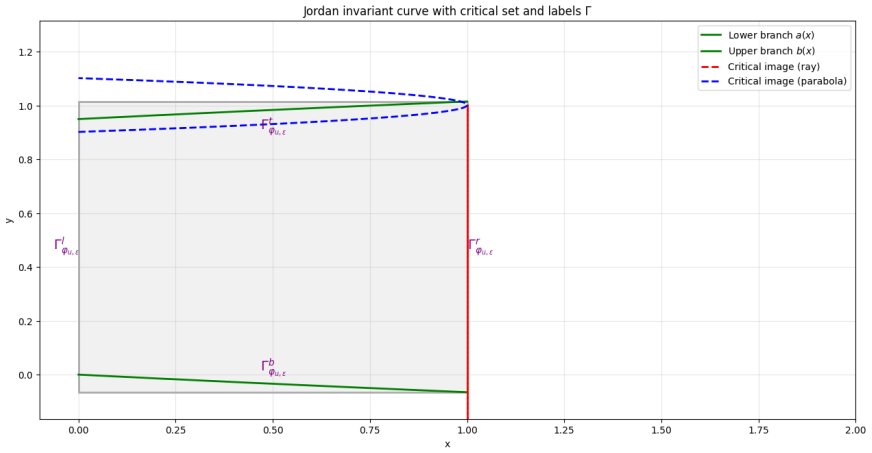



FIGURE 10. Approximation to the Jordan curve, $\mu = 4.0$ with its critical set, geometrically, the curve 'breaks': it remains continuous, but corners or folds appear.

## 5. Differentiability of the Limit Function

We devote this section to discussing some issues related to the differentiability of the limit function $\varphi_{\mu,\epsilon}$. The extreme points of $graph(\varphi_{\mu,\epsilon})$ are:

- $O = (0,0)$ and $O^1 = (1, -\frac{\epsilon}{\mu})$ when $\mu \geq 1 + \epsilon$; and
- $P_{\mu,\epsilon} = (0, \frac{\mu-\epsilon-1}{\mu})$ and $P^1_{\mu,\epsilon} = (1, \frac{1}{2\mu}(\mu - \epsilon - \sqrt{(\mu-\epsilon-2)^2 + 4\epsilon\mu}))$ when $\mu < 1+\epsilon$.

Recall that $a(0) = 0$ if $\mu \geq 1 + \epsilon$, and $a(0) = \frac{\mu-\epsilon-1}{\mu}$ when $\mu < 1 + \epsilon$. The eigenvalues of the differential $DF_{\mu,\epsilon}$ at the points $O$ and $P$ are:

$$\lambda_1(O) = \mu, \quad \lambda_2(O) = \mu - \epsilon,$$
$$\lambda_1(P_{\mu,\epsilon}) = \mu, \quad \lambda_2(P_{\mu,\epsilon}) = -\mu + \epsilon + 2.$$

**Remark 5.1.** *For $\lambda_2(O)$ and $\lambda_2(P_{\mu,\epsilon})$, the associated eigenspace is always the vertical line $\{x = 0\}$. The diagonal is the eigenspace corresponding to $\lambda_1(O)$ for all $\epsilon$ and $\mu$. That same line is the eigenspace associated to $\lambda_1(P_{\mu,\epsilon})$ if and only if $\mu = 1+\frac{\epsilon}{2}$; in this particular case $\lambda_1(P_{\mu,\epsilon}) = \lambda_2(P_{\mu,\epsilon})$ and $DF_{\mu,\epsilon}(P_{\mu,\epsilon})$ is non-diagonalizable. When $\mu \neq 1 + \frac{\epsilon}{2}$, the eigenspace relative to $\lambda_1(P_{\mu,\epsilon})$ is the line $y = \frac{\epsilon}{2\mu-\epsilon-2}x$.*

*It is worth noting that when $\mu = 1 + \epsilon$, the fixed points $O$ and $P_{\mu,\epsilon}$ not only coincide, they also lose hyperbolicity: $\lambda_2(O) = 1 = \lambda_2(P_{\mu,\epsilon})$. They exchange their hyperbolic nature across the parameter region separated by the line $\mu = 1 + \epsilon$: $O$ is repelling and $P_{\mu,\epsilon}$ is a hyperbolic saddle when $\mu > 1 + \epsilon$, while for $\mu < 1 + \epsilon$, $O$ is a hyperbolic saddle and $P_{\mu,\epsilon}$ is repelling. In other words, along the line $\mu = 1 + \epsilon$ the endomorphism $F_{\mu,\epsilon}$ undergoes a transcritical bifurcation at the origin $O$ (see [41][Chapter 3] for details.*

*As a consequence, for all $\mu, \epsilon$ with $\mu = 1 + \epsilon$, one can choose $r > 0$ sufficiently small such that every point $(x, y)$ with $|x| < r$ and $y \leq x$ has a pre-orbit $\{(x_n, y_n)\}_{n\geq 1}$*

*in the same region accumulating at $O$. That is, for each $n \geq 1$ there exists $(x_n, y_n)$ with $|x_n| < r$, $y_n \leq x_n$, and $F_{\mu,\epsilon}(x_n, y_n) = (x_{n-1}, y_{n-1})$, where $(x_0, y_0) = (x, y)$, such that $(x_n, y_n) \to O$ as $n \to +\infty$.*

*From this property and the fact that $(0, a(0))$ is repelling when $\mu \neq 1+\epsilon$, it follows that for all $\epsilon \in (0,1)$ and $1 < \mu \leq 4$, if $\varphi \in \mathcal{C}$ and $\varphi_n = \Phi^n(\varphi)$ is the $n$-th iterate of $\varphi$ under $\Phi$, then the graph$(\varphi_n)$ accumulates on the eigenspace associated to the least expanding eigenvalue at the fixed point $(0, a(0))$. Therefore graph$(\varphi_{\mu,\epsilon})$ is tangent at $(0, a(0))$ to that eigenspace. It is clear that these tangencies are expressed analytically by:*

- $\lim_{t \to 0^+} \dfrac{\varphi_{\mu,\epsilon}(t) - a(0)}{t} = -\infty$ *if* $\mu \geq 1 + \frac{\epsilon}{2}$, *and*

- $\lim_{t \to 0^+} \dfrac{\varphi_{\mu,\epsilon}(t) - a(0)}{t} = \frac{\epsilon}{2\mu - \epsilon - 2}$ *when* $\mu < 1 + \frac{\epsilon}{2}$.

*Complementary remarks to the previous discussion. Suppose that $\mu = 4$. It is well known that $\{f_4^{-n}(0) : n \geq 0\}$ is dense in the interval $[0,1]$. Therefore, the set of pre-orbits of the origin $\{F_{4,\epsilon}^{-n}(O) : n \geq 0\}$ is dense in* graph$(\varphi_{\mu,\epsilon})$. *Since* graph$(\varphi_{\mu,\epsilon})$ *has a vertical tangency at $O$ and this tangency is transported to each of its preimages, we conclude that there exists a dense subset of $[0,1]$ where $\varphi_{\mu,\epsilon}$ is not differentiable.*

*Another important observation: suppose that $\varphi \in C^1$ is differentiable at $f_\mu(s)$ for some $s \in (0,1)$. Then $\Phi(\varphi)$ is differentiable at any preimage of $s$ under $f_\mu$. In fact, from the very definition of the operator $\Phi$ we obtain*

$$(\Phi(\varphi))'(t) = \frac{\varphi'(f_\mu(s)) f'_\mu(s) - \epsilon}{\sqrt{(\mu - \epsilon)^2 + 4\epsilon\mu t - 4\mu\varphi(f_\mu(s))}}, \quad \text{if } f_\mu(t) = f_\mu(s). \tag{26}$$

*Graphical illustration of how* graph$(\varphi_\mu)$ *is positioned at $(0, a(0))$: on the right, the case when $\mu < 1 + \frac{\varepsilon}{2}$, and on the left, when $\mu \geq 1 + \frac{\varepsilon}{2}$.*

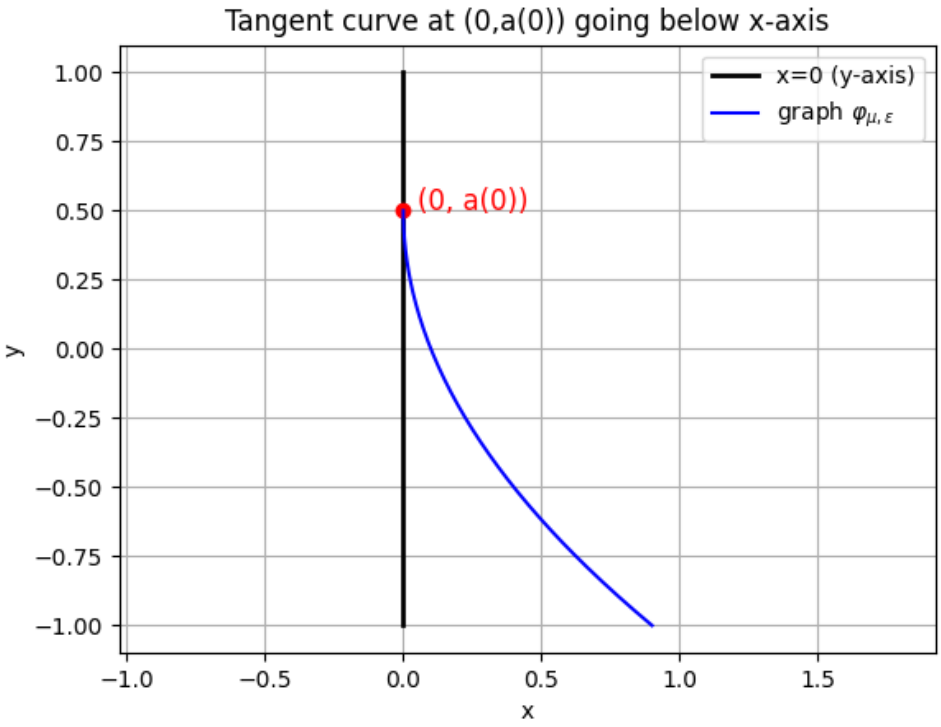



FIGURE 11. Graphical illustration of how graph$(\varphi_{\mu,\epsilon})$ is positioned at $(0, a(0))$: and on the left, when $\mu \geq 1 + \frac{\varepsilon}{2}$.

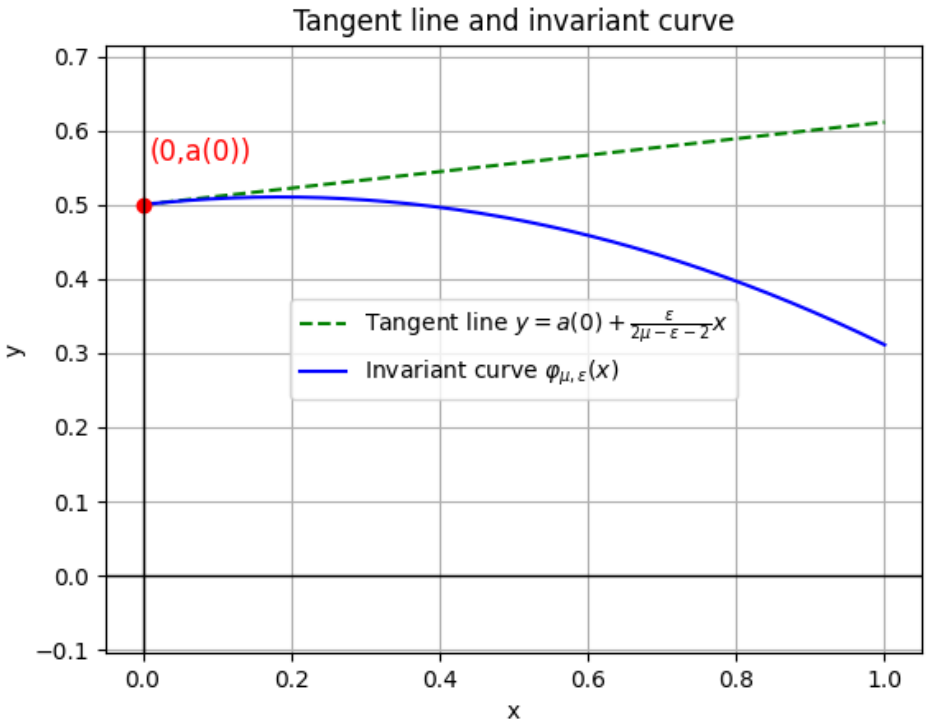



FIGURE 12. Graphical illustration of how graph$(\varphi_{\mu,\epsilon})$ is positioned at $(0, a(0))$: on the right, the case when $\mu < 1 + \frac{\varepsilon}{2}$.

*Moreover, by using* (19) *and* (20)*, the differentiability of* $\varphi_{\mu,\epsilon}$) *extends to other points. Consequently, if the limit function* $\varphi_{\mu,\epsilon}$ *were differentiable at every point of an interval* $[a, b] \subset (0, 1)$*, then it would also be differentiable at every point of any interval* $I \subset (0, 1)$ *such that* $f_\mu^n(I) = [a, b]$ *for some* $n \geq 1$.

*In our analysis of the differentiability of* $\varphi_{\mu,\epsilon}$ *we first consider the case* $1 < \mu \leq 3$*, only to associate the differentiability of* $\varphi_{\mu,\epsilon}$ *with the stable (resp. central) manifold of the fixed point* $R_{\mu,\epsilon}$ *if* $1 < \mu < 3$ *(resp.* $\mu = 3$*). Recall that this point belongs to* graph$(\varphi_{\mu,\epsilon})$*, and the eigenvalues of the differential* $DF_{\mu,\epsilon}(R_{\mu,\epsilon})$ *are* $2 - \mu$ *and* $\mu + \epsilon$*, with respective eigenspaces given by the line* $y = \frac{\epsilon}{2-2\mu-\epsilon}x$ *and the vertical* $x = 0$*. Note that* $2 - 2\mu - \epsilon \neq 0$ *and* $\frac{|2-\mu|}{\mu+\epsilon} < 1$*. Consequently, for* $1 < \mu < 3$ *the point* $R_{\mu,\epsilon}$ *is a hyperbolic saddle with expansion along the vertical direction. When* $\mu = 3$ *this point loses hyperbolicity but preserves vertical expansion.*

*On the other hand, for* $1 < \mu \leq 3$ *the fixed point* $\frac{\mu-1}{\mu}$ *of* $f_\mu$ *satisfies* $f_\mu^n(t) \to \frac{\mu-1}{\mu}$ *for all* $t \in (0, 1)$ *as* $n \to +\infty$*. The positive invariance of* graph$(\varphi_{\mu,\epsilon})$ *under* $F_{\mu,\epsilon}$ *implies that* $F_{\mu,\epsilon}^n(t, \varphi(t)) \to R_{\mu,\epsilon}$ *as* $n \to +\infty$*, for any* $t \in (0, 1)$*. Thus, the curve* graph$(\varphi_{\mu,\epsilon})$ *restricted to* $(0, 1)$ *is the connected component containing* $R_{\mu,\epsilon}$ *of the stable manifold* $W^s(R_{\mu,\epsilon})$ *of* $R_{\mu,\epsilon}$*; while if* $\mu = 3$*, that same piece of* graph$(\varphi_{\mu,\epsilon})$ *is the connected component of the central manifold* $W^c(R_{\mu,\epsilon})$ *containing this point. This proves the differentiability, in fact* $C^\infty$*, of* $\varphi_{\mu,\epsilon}$ *in* $(0, 1)$ *when* $1 < \mu \leq 3$.

*Following the previous discussion, it is clear that the differentiability of $\varphi_{\mu,\epsilon}$ extends to the endpoints of $(0,1)$ only in the case $\mu < 1 + \frac{\epsilon}{2}$.*

**Theorem 5.1.** *For each $\epsilon \in (0,1)$ and $1 < \mu \leq 3$ the function $\varphi_{\mu,\epsilon}$ is differentiable in $(0,1)$ (indeed $C^\infty$); this property extends to the endpoints of $(0,1)$ if and only if $\mu < 1 + \frac{\epsilon}{2}$.*

*To continue the analysis of the differentiability of $\varphi_{\mu,\epsilon}$ when $3 < \mu < 4$, two options may be considered.*

**Option 1.** *Let $(\mu_n)_{n\geq 0}$ be the sequence of parameter values $\mu$ that define the period-doubling cascade unfolded by $f_\mu$ at the fixed point $x = \frac{\mu-1}{\mu}$, starting from $\mu_0 = 3$. Recall that $\mu_n \nearrow \mu_\infty$ as $n \to +\infty$, where $\mu_\infty \approx 3.56995$ (see [40]). In other words, for each $n \geq 1$ and every $\mu \in (\mu_{n-1}, \mu_n)$, the map $f_\mu$ admits an attracting $2^n$-periodic orbit*

$$P_n = \{p_j^n : j = 0, \ldots, 2^n - 1\},$$

*satisfying $f_\mu^n(p_j^n) = p_{j+1}^n$ for $j < 2^n - 1$, $f_\mu^n(p_{2^n-1}^n) = p_0^n$, and $|(f_\mu^n)'(p_j^n)| < 1$ for all $j = 0, \ldots, 2^n - 1$. Moreover, the $\omega$-limit set of every point in $(0,1)$ is $P_n$, except for those that are preimages of periodic points of smaller period $2^k$, with $k = 0, \ldots, n-1$, which are repelling.*

*From these periodic orbits $P_n$, the invariance of $graph(\varphi_{\mu,\epsilon})$, and the dynamics of the restriction $F_{\mu,\epsilon}(graph(\varphi_{\mu,\epsilon})$, it follows that for each $n \geq 1$ and every $\mu \in (\mu_{n-1}, \mu_n)$, the endomorphism $\mathcal{F}_\mu$ admits a $2^n$-periodic orbit*

$$\widehat{P}_n = \{(p_j^n, \varphi_{\mu,\epsilon}(p_j^n)) : j = 0, \ldots, 2^n - 1\},$$

*with the following properties:*

*a) $\widehat{P}_n$ is a hyperbolic saddle. At each point $q_j^n = (p_j^n, \varphi_{\mu,\epsilon}(p_j^n))$, $j = 0, \ldots, 2^n - 1$, the differential of $F_{\mu,\epsilon}^n$ has eigenvalues*

$$(f_\mu^n)'(p_j^n) = \prod_{i=0}^{n-1} f_\mu'(p_i^n) \text{ and } \quad \prod_{i=0}^{n-1} \big(f_\mu'(\varphi_{\mu,\epsilon}(p_i^n)) - \epsilon\big).$$

*Note that $f_\mu'(\varphi_{\mu,\epsilon}(p_i^n)) \geq \mu$ since $\varphi_{\mu,\epsilon}(p_i^n) \leq 0$ for all $i = 0, \ldots, 2^n - 1$. Moreover, $\widehat{P}_n$ is the unique $2^n$-periodic Moreover, $\widehat{P}_n$ is the unique $2^n$-periodic orbit of $F_{\mu,\epsilon}$ in $graph(\varphi_{\mu,\epsilon})$, and for each $k = 0, \ldots, n-1$, $\widehat{P}_k$ is the only periodic orbit of $F_{\mu,\epsilon}(graph(\varphi_{\mu,\epsilon})$ with period different from $2^n$, all of which are repelling.*

*b) For every $n \geq 1$, $\mu \in (\mu_{n-1}, \mu_n)$, and $j = 0, \ldots, 2^{n-1}$, there exists an open interval $I_j^n \subset (0,1)$ with $p_j^n \in I_j^n$ such that $\{(t, \varphi_{\mu,\epsilon}(t)) : t \in I_j^n\}$ is the local stable manifold $W_{loc}^s(q_j^n)$ of $q_j^n$ (the connected component of $W^s(q_j^n)$ containing $q_j^n$). This property extends to $\mu = \mu_n$, where $\widehat{P}_n$ loses hyperbolicity and each segment $\{(t, \varphi_{\mu,\epsilon}(t)) : t \in I_j^n\}$ becomes the local center manifold $W_{loc}^c(q_j^n)$. Note that the endpoints of the intervals $I_j^n$ lie in the orbits $P_k$ ($k < n$) or in some of their preimages.*

*As a consequence of these two properties, $\varphi_{\mu,\epsilon}$ is differentiable (indeed $C^\infty$) on each $I_j^n$ and also on their preimages (see the remarks above concerning (19) and (20)). Thus, to achieve differentiability of $\varphi_{\mu,\epsilon}$ throughout the interval $(0,1)$, it is necessary that the tangent bundle of $graph(\varphi_{\mu,\epsilon})|_{I_j^n}$ (the local stable/center manifold*

*of $q_j^n$) be continuous at the endpoints of each $I_j^n$. This occurs only if, at those endpoints, the differential of suitable iterates of $F_{\mu,\epsilon}$ expands more strongly in the vertical direction than in the other eigenspace. This can be verified without difficulty in the case $n = 1$.*

*Take any $\epsilon \in (0,1)$ and $\mu \in (3, \mu_1]$, let $P_1 = \{p_0^1, p_1^1\}$ be the attracting 2-periodic orbit of $f_\mu$ described above; clearly $p_0^1 < \frac{\mu-1}{\mu} < p_1^1$. Let $r_0 < \frac{\mu-1}{\mu} < r_1$ be the immediate preimages under $f_\mu^2$ of $\frac{\mu-1}{\mu}$. Then $I_0^1 = (r_0, \frac{\mu-1}{\mu})$ and $I_1^1 = (\frac{\mu-1}{\mu}, r_1)$ are the intervals such that*

$$graph(f_\mu)|_{I_j^1} = W_{loc}^{s(c)}(q_j^1) = \{(t, f_\mu(t)) : t \in I_j^1\}, \quad j = 0, 1.$$

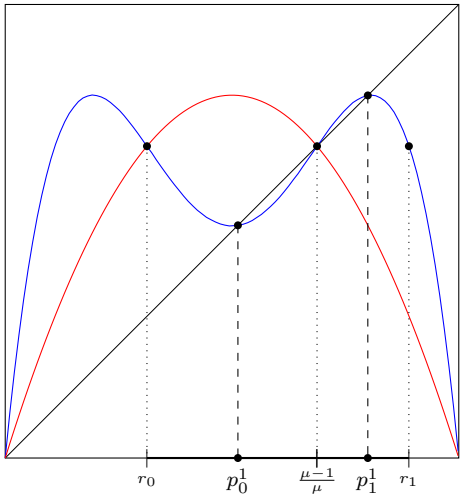



FIGURE 13. Numerical illustration of the graphs of the logistic function and its square for $\mu = 3.2$, showing the attracting 2-periodic orbit $\{p_0^1, p_1^1\}$ and the connected components of the stable manifold containing each of them..

*On the other hand, since in $R_{\mu,\epsilon}$ the differential of $F_{\mu,\epsilon}$ is*

$$\begin{pmatrix} 2-\mu & 0 \\ \epsilon & \mu+\epsilon \end{pmatrix},$$

*and obviously $\mu + \epsilon > |2 - \mu|$, its inverse $B$ is such that, for any nonzero and nonvertical vector $\begin{pmatrix} u \\ v \end{pmatrix}$, the slope of the vector $B^n \begin{pmatrix} u \\ v \end{pmatrix}$ is given by*

$$\text{slope}\left(B^n \begin{pmatrix} u \\ v \end{pmatrix}\right) = \frac{-\epsilon \sum_{k=0}^{n-1} (\mu+\epsilon)^{n-1-k} (2-\mu)^k}{(\mu+\epsilon)^n} + \frac{v}{u} \left(\frac{\mu+\epsilon}{2-\mu}\right)^n, \quad \forall n \geq 1,$$

*from which it follows that*

$$\text{slope}\left(B^n \begin{pmatrix} u \\ v \end{pmatrix}\right) \to \frac{\epsilon}{2-2\mu-\epsilon} \quad \text{as } n \to +\infty.$$

*This implies the differentiability of* $\varphi_{\mu,\epsilon}$ *at* $\frac{\mu-1}{\mu}$ *(recall that* $y = \frac{\epsilon}{2-2\mu-\epsilon}x$ *defines the eigenspace associated with* $\lambda_1 = 2-\mu$*), and therefore Theorem 5.1 extends to* $1 < \mu \leq \mu_1$ *and* $\epsilon \in (0,1)$.

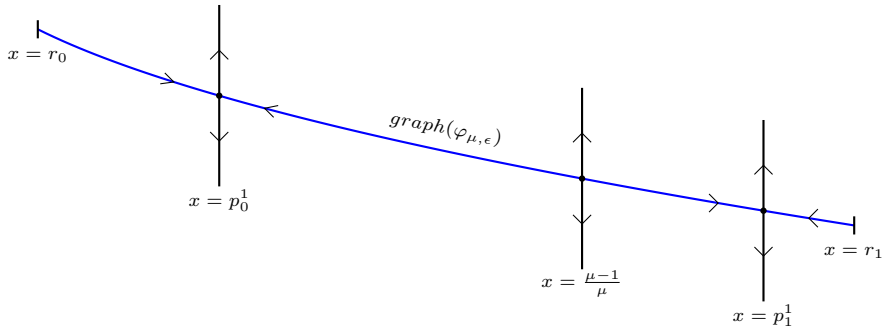



FIGURE 14. Translation to $graph(\varphi_{\mu,\epsilon})$ of the situation described for $f_\mu$ and the 2-periodic orbit $P_1$.

*One could continue along this line of arguments and use recurrence to attempt to show the differentiability of* $\varphi_\mu$ *in* $(0,1)$ *when* $\mu \in (\mu_{n-1}, \mu_n]$ *and* $\epsilon \in (0,1)$ *with* $n \geq 2$*; although for* $n = 2$ *the calculations are simple, for* $n > 2$ *they become difficult to manipulate algebraically, particularly because they involve computing roots of polynomials of degree greater than or equal to* 8*. Precisely, the simplicity of the case* $n = 2$ *is supported by the precise knowledge of the* 2*-periodic orbit* $P_1$ *and of* $(f_\mu^2)'(p_{1j})$, $j = 0, 1$*:*

$$p_{10},\ p_{11} = \frac{1}{2\mu}\left(\mu + 1 \pm \sqrt{(\mu+1)(\mu-3)}\right), \qquad (f_\mu^2)'(p_{1j}) = 4 + 2\mu - \mu^2.$$

*Thus, to see that*

$$\left|4 + 2\mu - \mu^2\right| < \left(f'_\mu(\varphi_{\mu,\epsilon}(p_{10})) - \epsilon\right)\left(f'_\mu(\varphi_{\mu,\epsilon}(p_{12})) - \epsilon\right) \quad \text{if } \mu > \mu_1 = 1 + \sqrt{6},$$

*it suffices to observe that*

$$\left|4 + 2\mu - \mu^2\right| < 4 \quad \text{for each } 3 \leq \mu < 4,$$

*and that*

$$f'_\mu(\varphi_{\mu,\epsilon}(p_{1j})) > 4 \quad \text{for all } 1 < \mu < 4,\ \epsilon \in (0,1).$$

*Recall that* $\varphi_{\mu,\epsilon}(t) < 0$ *for all* $t \in (0,1)$.

*In our opinion, this procedure is inappropriate due to the complexity of the calculations, and even more so if one intends to apply it for values of* $\mu$ *greater than* $\mu_\infty$.

*A procedure that could be employed is in fact a classical one; it is a sophisticated version, in the context of tangent bundles, of Banach's contraction theorem, known as the* Contraction Theorem on Fibers. *This approach was used in the remarkable and fruitful work of M. Hirsch, C. Pugh, and M. Shub [2] to establish the differentiability of invariant manifolds; see also [5]. The version in metric spaces that we reproduce here, including its proof, appears in [39].*

**Theorem 5.2 (Contraction on Fibers).** *Let $(X,\rho)$ and $(Y,\delta)$ be complete metric spaces and let $F: X\times Y \to X\times Y$ be an operator of the form $F(x,y)=(f(x),g(x,y))$. Suppose that the maps $f: X\to X$ and $g: X\times Y\to Y$ satisfy:*

*a) The operator $f$ admits a fixed point $x_0\in X$ which is a global attractor; that is, $f(x_0)=x_0$ and $\rho(f^n(x),x_0)\to 0$ as $n\to+\infty$.*

*b) For each $y\in Y$, the operator $g_y: X\to X$ is continuous, where $g_y(x)=g(x,y)$ for all $x\in X$.*

*c) There exists $0<\lambda<1$ such that for each $x\in X$, the operator $g^x: Y\to Y$ defined by $g^x(y)=g(x,y)$ is a $\lambda$-contraction.*

*Then, for the attracting fixed point $y_0\in Y$ of the operator $g^{x_0}$, the pair $(x_0,y_0)$ is a fixed point of $F$, and $F^n(x,y)\to(x_0,y_0)$ as $n\to+\infty$, for every $(x,y)\in X\times Y$.*

*If one wishes to follow the classical ideas around the application of this result to show differentiability properties, then the following are required:*

*1) A space $(X,\rho)$ where the function to be differentiated "lives", in our case $\varphi_{\mu,\epsilon}$. We already have such a space: $\mathcal{C}$ as defined above, or perhaps a refinement of it that somehow optimizes the region $R$. In any case, the operator $f$ on $X$ is also defined, namely $\Phi:\mathcal{C}\to\mathcal{C}$ described in (18). With such a space and operator, hypothesis (a) in Theorem 5.2 is satisfied.*

*2) A space $(Y,\delta)$ corresponding to continuous functions used to describe the derivatives of those $C^1$ functions in $\mathcal{C}$. In attempting to model this situation we encounter a slight obstacle: completeness. We know that if $\mu>1+\frac{\epsilon}{2}$, the function $\varphi_{\mu,\epsilon}$ is not differentiable at the endpoints of $[0,1]$; therefore this interval cannot be the domain of the functions in $Y$. Suppose then that the functions in $Y$ have domain $[a,b]\subset(0,1)$; with the usual metric, $(Y,\delta)$ is complete. We also know that if $\varphi\in\mathcal{C}$ is differentiable on $[a,b]$, then $\Phi(\varphi)$ is differentiable at every point $s\in(0,1)$ such that $f_\mu(s)\in[a,b]$; that is, $s\in f_\mu^{-1}([a,b])$, see (26). Thus, to make this compatible one must have $f_\mu([a,b])\subset[a,b]$. Note also that if differentiability is desired on $(0,1)$, the interval $[a,b]$ must be such that $\bigcup_{n\geq 0} f_\mu^{-n}([a,b])=(0,1)$.*

*3) Assuming $(Y,\delta)$ is defined as the space of continuous functions $L:[a,b]\to\mathbb{R}$ with $f_\mu([a,b])\subset[a,b]$ and the usual metric, it remains to define $g: X\times Y\to Y$. The natural way to do this, according to (26), is to consider, for each $\varphi\in\mathcal{C}$, $L\in Y$, and $s\in[a,b]$:*

$$g^\varphi(L)(s)=g(\varphi,L)(s)=\frac{L(f_\mu(s))f'_\mu(s)-\epsilon}{\sqrt{(\mu-\epsilon)^2+4\epsilon\mu s-4\mu\varphi(f_\mu(s))}}. \tag{27}$$

*The first observation is the necessity of $f_\mu([a,b])\subset[a,b]$; note also that for each $L\in Y$, the partial function $g_L: X\to X$ is continuous, so hypothesis (b) of Theorem 5.2 is also satisfied. The crucial step is then to show the third requirement in the theorem. Suppose we fix $\varphi\in\mathcal{C}$; then for each $L,G\in Y$ and each $s\in[a,b]$ we have*

$$|g^\varphi(L)(s)-g^\varphi(G)(s)|\leq\frac{|f'_\mu(s)|}{\sqrt{(\mu-\epsilon)^2+4\epsilon\mu s-4\mu\varphi(f_\mu(s))}}\delta(L,G), \tag{28}$$

*which forces us to bound above the quotient $\frac{|f'_\mu(s)|}{\sqrt{(\mu-\epsilon)^2+4\epsilon\mu s-4\mu\varphi(f_\mu(s))}}$. It should come as no surprise that in attempting to obtain an upper bound for this quotient not exceeding $1$, we encounter a situation of the same nature as discussed in*

*Option 1. Indeed, from the expressions for* $\Phi(\varphi(s))$ *and* $f'_\mu(\Phi(\varphi(s)))$*, it follows that*

$$\sqrt{(\mu-\epsilon)^2+4\epsilon\mu s-4\mu\varphi(f_\mu(s))}=f'_\mu(\Phi(\varphi(s)))-\epsilon;$$

*since* $DF_{\mu,\epsilon}(s,\Phi(\varphi(s)))=\begin{pmatrix} f'_\mu(s) & 0 \\ \epsilon & f'_\mu(\Phi(\varphi(s)))-\epsilon\end{pmatrix}$*, hypothesis (c) in Theorem 5.2 is satisfied only if the differential* $DF_{\mu,\epsilon}(s,\Phi(\varphi(s)))$ *expands more along the vertical* $\{x=0\}$ *than along its other eigenspace, the one associated with* $f'_\mu(s)$*.*

*Suppose for a moment that we have* $[a,b]\subset(0,1)$*,* $\mu^*\in(1,4)$ *and* $\epsilon^*\in(0,1)$ *such that, for* $\mu\in(1,\mu^*)$ *and* $\epsilon\in(0,\epsilon^*)$*, there exists* $\lambda\in(0,1)$ *(possibly depending on* $\mu$ *and* $\epsilon$*) in such a way that for every* $\varphi\in\mathcal{C}$ *and each* $s\in[a,b]$ *one has*

$$\frac{|f'_\mu(s)|}{\sqrt{(\mu-\epsilon)^2+4\epsilon\mu s-4\mu\varphi(f_\mu(s))}}\leq\lambda;$$

*that is,* $g^\varphi:Y\to Y$ *is a* $\lambda$*-contraction (uniform in* $\varphi$*). Then, if* $F$ *is as in the statement of Theorem 5.2 and* $L_0$ *is the fixed point of* $g^{\varphi_{\mu,\epsilon}}$*, we have* $F^n(\varphi,L)\to(\varphi_{\mu,\epsilon},L_0)$ *as* $n\to+\infty$*, for all* $(\varphi,L)\in X\times Y$*. Since*

$$F^n(\varphi,L)=(\Phi^n(\varphi),(g^{\Phi^{n-1}(\varphi)}\circ\cdots\circ g^\varphi)(L))\quad\text{for each } n\geq1,$$

*it is clear that* $\Phi^n(\varphi)$ *and* $(g^{\Phi^{n-1}(\varphi)}\circ\cdots\circ g^\varphi)(L)$ *converge respectively and uniformly to* $\varphi_{\mu,\epsilon}$ *and* $L_0$ *on* $[0,1]$ *and* $[a,b]$ *as* $n\to+\infty$*. In particular, if* $\varphi\in\mathcal{C}$ *is differentiable on* $[a,b]$ *and* $L=\varphi'$*, then* $\Phi(\varphi)$ *is differentiable (see* (26)*, recall that* $f_\mu([a,b])\subset[a,b]$*) and from the definition of* $g^\varphi$ *we have* $(\Phi(\varphi))'=g^\varphi(\varphi')$*; indeed, by induction one shows that* $\Phi^n(\varphi)$ *is differentiable and*

$$(\Phi^n(\varphi))'=(g^{\Phi^{n-1}(\varphi)}\circ\cdots\circ g^\varphi)(\varphi')$$

*for each* $n\geq1$*. In this way* $\Phi^n(\varphi)\to\varphi_{\mu,\epsilon}$ *uniformly on* $[0,1]$ *and* $(\Phi^n(\varphi))'$ *converges to* $L_0$ *uniformly on* $[a,b]$*, therefore the function* $\varphi_{\mu,\epsilon}$ *is differentiable on* $[a,b]$ *and moreover* $\varphi'_{\mu,\epsilon}=L_0$*.*

*To make use of Theorem 5.2 , we first perform a small refinement in the complete metric space* $\mathcal{C}$*. Since the endpoints of* $graph(\varphi_{\mu,\epsilon})$ *are* $O$ *and* $O^1$*, we consider the line segment of equation* $y=-\frac{\epsilon}{\mu}x$ *in the interval* $[0,1]$*; the preimage of this segment is given by*

$$y_0(t)=\tfrac{1}{2\mu}\left(\mu-\epsilon-\sqrt{(\mu+\epsilon)^2-4\epsilon\mu(t-1)^2}\right)\quad\text{for each } t\in[0,1].$$

*It is easy to check that* $y_0(0)=0$*,* $y_0(1)=-\frac{\epsilon}{\mu}$ *and* $\Phi(\varphi_0)(t)<y_0(t)<\varphi_0(t)$ *for all* $t\in(0,1)$*; recall that* $\varphi_0$ *is the function whose graph defines the curve* $\gamma_0$ *in the definition of the region* $R$*. This leads us, in order to facilitate calculations, to consider* $\mathcal{C}$ *as the space of all continuous functions* $\varphi:[0,1]\to\mathbb{R}$ *such that* $\varphi(0)=0$*,* $\varphi(1)=-\frac{\epsilon}{\mu}$*,* $graph(\varphi)\subset R$*, and* $\varphi(t)\leq y_0(t)$ *for each* $t\in[0,1]$*. Obviously, with these adjustments no changes occur either in the definition of the operator* $\Phi$ *or in the conclusion of Lemma 4.2.*

*Now we introduce the complete metric space* $(\widetilde{\mathcal{C}},\delta)$ *of all continuous functions* $L:\left[\frac{4-\mu}{4},\frac{\mu}{4}\right]\to\mathbb{R}$ *with*

$$\delta(L,G)=\max\{|L(t)-G(t)|:t\in\left[\tfrac{4-\mu}{4},\tfrac{\mu}{4}\right]\},$$

*and we define the operator*

$$\Psi : \mathcal{C} \times \widetilde{\mathcal{C}} \to \mathcal{C} \times \widetilde{\mathcal{C}}$$

*by*

$$\Psi(\varphi, L) = (\Phi(\varphi), \widehat{\Phi}(\varphi, L)), \quad (\varphi, L) \in \mathcal{C} \times \widetilde{\mathcal{C}}, \tag{29}$$

*where* $\widehat{\Phi} : \mathcal{C} \times \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}$ *is given, for each* $t \in \left[\frac{4-\mu}{4}, \frac{\mu}{4}\right]$, *by*

$$\widehat{\Phi}(\varphi, L)(t) = \frac{L(f_\mu(t))f'_\mu(t) - \epsilon}{\sqrt{(\mu-\epsilon)^2 + 4\epsilon\mu t - 4\mu\varphi(f_\mu(t))}}. \tag{30}$$

*Since* $f_\mu(t) \in \left[\frac{4-\mu}{4}, \frac{\mu}{4}\right]$ *for each* $t \in \left[\frac{4-\mu}{4}, \frac{\mu}{4}\right]$, $\widehat{\Phi}(\varphi, L)$ *is indeed a function in* $\widetilde{\mathcal{C}}$. *Note also that if* $\varphi \in \mathcal{C}$ *is of class* $C^1$ *and* $L = \varphi'$ *on* $\left[\frac{4-\mu}{4}, \frac{\mu}{4}\right]$, *then the expression above coincides with* (26). *On the other hand, if in the statement of Theorem 5.2 we take* $X = \mathcal{C}$, $Y = \widetilde{\mathcal{C}}$ *and* $F = \Psi$, *then conditions (a) and (b) are satisfied; let us now show that condition (c) also holds.*

*Let* $\varphi \in \mathcal{C}$, $L, G \in \widetilde{\mathcal{C}}$ *and* $t \in \left[\frac{4-\mu}{4}, \frac{\mu}{4}\right]$. *Then it is clear that:*

$$|\widehat{\Phi}(\varphi, L)(t) - \widehat{\Phi}(\varphi, G)(t)| \leq \frac{|f'_\mu(t)|}{\sqrt{(\mu-\epsilon)^2 + 4\epsilon\mu t - 4\mu\varphi(f_\mu(t))}}\delta(L, G). \tag{31}$$

*Using the definition of* $\Phi$, *we note that*

$$\sqrt{(\mu-\epsilon)^2 + 4\epsilon\mu t - 4\mu\varphi(f_\mu(t))}$$

*is precisely* $f'_\mu(\Phi(\varphi)(t)) - \epsilon$. *Now, since* $\Phi(\varphi)(t) \leq y(t) \leq 0$ *for each* $t \in \left[\frac{4-\mu}{4}, \frac{\mu}{4}\right]$, *we have* $f'_\mu(\Phi(\varphi)(t)) \geq f'_\mu(y_0(t))$, *and therefore the inequality*

$$\sqrt{(\mu-\epsilon)^2 + 4\epsilon\mu t - 4\mu\varphi(f_\mu(t))} \geq \sqrt{(\mu+\epsilon)^2 - 4\epsilon\mu(t-1)^2}, \quad \text{for all } t \in \left[\frac{4-\mu}{4}, \frac{\mu}{4}\right].$$

*It is straightforward to verify, for each* $t \in \left[\frac{4-\mu}{4}, \frac{\mu}{4}\right]$, *the bounds*

$$|f'_\mu(t)| \leq \tfrac{\mu}{2}(\mu - 2) \quad \text{and} \quad \sqrt{(\mu+\epsilon)^2 - 4\epsilon\mu(t-1)^2} \geq \tfrac{1}{2}\sqrt{4(\mu+\epsilon)^2 - \epsilon\mu^3}.$$

*From these and* (31) *it follows that*

$$\delta(\widehat{\Phi}^\varphi(L), \widehat{\Phi}^\varphi(G)) \leq \frac{2\mu}{\sqrt{4(\mu+\epsilon)^2 - \epsilon\mu^3}}\delta(L, G).$$

*Finally, since*

$$4(\mu+\epsilon)^2 - \epsilon\mu^3 = 4\mu^2 + 8\epsilon\mu + 4\epsilon^2 - \epsilon\mu^3 > 4\mu^2,$$

*the desired contraction property is established.*

## 6. Acknowledgments

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