Homoclinic Holonomy and Livšic Obstructions in Anosov Extensions

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Abstract. We study infinitesimal aspects of isometric extensions of Anosov flows from the viewpoint of homoclinic holonomy and Livšic cohomology. In the Brin–Pesin–Lefeuvre framework, invariant functions and invariant sections are controlled by a transitivity representation associated with homoclinic holonomies, and the zero Pollicott–Ruelle resonance detects smooth flow-invariant sections [BP74, Lef23]. We collect three existing threads—Brin–Pesin–Lefeuvre transitivity, the Livšic cohomological equation and Pollicott–Ruelle resonances—into a deformation-theoretic dictionary, and formulate questions about their interaction. The corresponding first-order deformation problem leads to a twisted Livšic cohomological equation: a perturbation of a cocycle is trivial up to gauge when it solves this equation, so periodic orbit data provide the natural obstruction theory [Liv72, PP90, PP97, Wal00]. In the abelian case, closed one-forms define a finite-dimensional topological slice of the Livšic quotient, while the full Livšic quotient remains generally infinite-dimensional. This topological slice is compatible with the de Rham interpretation of zero Pollicott–Ruelle resonances in the sense of Dang–Rivière [DR20]. We also discuss how finite jets of Poincaré return maps near periodic orbits connect with Sternberg-type and non-stationary normal-form obstructions [Ste57, GK98, KS16, Kal20], suggesting a bridge between homoclinic holonomy, microlocal resonances and jet-theoretic normal forms.

Setup and known input. Let (M, φ_t) be a smooth volume-preserving Anosov flow with generator X, and let $\pi : E \to M$ be an isometric extension with compact fiber F. For an associated vector bundle $\mathcal{E} \to M$, the lifted flow induces a transport operator

$$\mathbf{X} : C^\infty(M, \mathcal{E}) \longrightarrow C^\infty(M, \mathcal{E}), \qquad \mathbf{X}(fu) = (Xf)u + f\mathbf{X}u.$$

Homoclinic stable/unstable holonomies generate a transitivity representation ρ of Parry's homoclinic monoid. If $H_\star$ denotes the corresponding closed transitivity group, Lefeuvre's microlocal formulation of the Brin–Pesin mechanism identifies invariant sections with fixed vectors of this representation [Lef23]. In particular, for an adjoint bundle,

$$\ker \mathbf{X}_{\mathrm{ad}\, P} \;\cong\; \mathfrak{g}^{H_\star} = \mathfrak{z}_{\mathfrak{g}}(H_\star),$$

so infinitesimal symmetries are centralizers of the homoclinic transitivity group. This part is a reformulation of known structural results, not a new theorem claimed here.

First-order deformation problem. Locally, a G-valued isometric cocycle may be represented infinitesimally by a generator $a : M \to \mathfrak{g}$, with twisted transport

$$\mathbf{X}_a u = Xu + [a, u].$$

For a perturbation $a_\varepsilon = a + \varepsilon b + O(\varepsilon^2)$, an infinitesimal gauge change $u : M \to \mathfrak{g}$ acts by

$$b \;\sim\; b + \mathbf{X}_a u.$$

Thus, in the Livšic–Parry–Pollicott–Walkden framework [Liv72, PP90, PP97, Wal00], the first-order deformation quotient is naturally the twisted Livšic quotient

$$H^1_{\mathrm{Liv}}(X, \mathrm{ad}\, P) := C^\infty(M, \mathrm{ad}\, P) \big/ \mathbf{X}_a C^\infty(M, \mathrm{ad}\, P).$$

This quotient should not be identified with de Rham cohomology in general; it is typically large. Its obstruction theory is encoded by periodic orbit data: if $b = \mathbf{X}_a u$, then the corresponding twisted periodic integrals vanish along every closed orbit $\gamma \in \mathrm{Per}(\varphi_t)$, in the spirit of the Livšic theory for hyperbolic flows and compact Lie group extensions [Liv72, PP97, Wal00].

Topological slice. In the abelian case, closed one-forms give a finite-dimensional topological slice. If $\alpha \in \Omega^1(M)$ is closed, then

$$b_\alpha = \alpha(X)$$

defines a Livšic class; if $\alpha = du$, then $b_\alpha = Xu$. Hence there is a natural map

$$H^1_{\mathrm{dR}}(M; \mathbb{R}) \longrightarrow H^1_{\mathrm{Liv}}(X), \qquad [\alpha] \longmapsto [\alpha(X)].$$

For $M = S^*\Sigma$ with Σ a closed surface of genus g, this slice has dimension $2g$. This is only a topological slice, not the full Livšic quotient. The point of this distinction is to avoid conflating the generally infinite-dimensional Livšic quotient

with the finite-dimensional de Rham complex. Dang–Rivière's theorem gives a different but compatible realization of twisted de Rham cohomology through invariant Pollicott–Ruelle resonant states [DR20].

Periodic jets and microlocal filtration. For each periodic orbit γ, a Poincaré return map

$$P_\gamma : (\Sigma_\gamma, x_\gamma) \to (\Sigma_\gamma, x_\gamma)$$

carries finite jet data $j^r P_\gamma$. These jets encode local Sternberg normal-form obstructions near hyperbolic periodic data [Ste57]. In the non-stationary setting, normal forms along contracting foliations and contracting extensions were developed by Guysinsky–Katok, Kalinin–Sadovskaya and Kalinin [GK98, KS16, Kal20]. We propose to compare this local jet-level information with Livšic periodic obstructions and with the zero-resonance/invariant-sector picture above.

A further question is whether band structures for twisted Pollicott–Ruelle resonances provide a useful filtration of this deformation picture. Here the intended input is not an identification theorem. In the locally symmetric case, such a filtration should be made precise using Küster–Weich's exact band structures for associated vector bundles over rank-one locally symmetric spaces [KW21]. The genuinely open direction is to understand how much of this picture survives for variable-curvature/contact Anosov flows, where Faure–Tsujii establish vertical band structures in the scalar/contact setting [FT13].

Status of the proposal.

- *Known structural input:* Brin–Pesin–Lefeuvre transitivity groups, invariant sections and zero Pollicott–Ruelle resonances [BP74, Lef23].
- *Formal deformation input:* first-order gauge equivalence leads to a twisted Livšic equation $\mathbf{X}_a u = b$.
- *Safe finite-dimensional slice:* closed one-forms give a de Rham slice inside the Livšic quotient; this is not the full quotient.
- *Proposed direction:* compare periodic Livšic data, Sternberg/non-stationary jets of Poincaré maps and microlocal band structures.

Guiding questions.

- How does the transitivity group $H_\star$ interact with the twisted Livšic quotient $H^1_{\mathrm{Liv}}(X, \operatorname{ad} P)$?
- When do pinching, semisimplicity or strong transitivity force a small "rigid window"?
- Can Küster–Weich-type band structures make this filtration precise in locally symmetric cases, and what remains true in variable curvature?

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